

# THE $\kappa$ -WORD PROBLEM OVER DRH

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**ABSTRACT.** Let  $\mathbf{H}$  be a pseudovariety of groups in which the  $\kappa$ -word problem is decidable. Here,  $\kappa$  denotes the canonical implicit signature, which consists of the multiplication and the  $(\omega - 1)$ -power. We prove that the  $\kappa$ -word problem is also decidable over  $\mathbf{DRH}$ , the pseudovariety of all finite semigroups whose regular  $\mathcal{R}$ -classes lie in  $\mathbf{H}$ . Further, we present a canonical form for elements in the free  $\kappa$ -semigroup over  $\mathbf{DRH}$ , based on the knowledge of a canonical form for elements in the free  $\kappa$ -semigroup over  $\mathbf{H}$ . This extends work of Almeida and Zeitoun on the pseudovariety of all finite  $\mathcal{R}$ -trivial semigroups.

## 1. INTRODUCTION

Decidability of word problems has driven researchers' attention for many years. Generally speaking, it consists in finding an algorithm (or disprove its existence) to test whether two representations of elements of a given structure define the same element. On the other hand, pseudovarieties play an essential role since Eilenberg's correspondence was formulated in 1976 [8, Chapter VII, Theorem 3.4s]. He showed that pseudovarieties of finite semigroups are in a bijective correspondence with varieties of rational languages. In turn, the study of the latter is strongly motivated by its application in Computer Science, namely, through the study of Automata Theory. For that reason, deciding whether two given expressions have the same value over all semigroups in a certain pseudovariety seems to be a relevant question. Besides these motivations, solving the word problem for  $\sigma$ -words (with  $\sigma$  an implicit signature) over a pseudovariety  $\mathbf{V}$  appears as an intermediate step to prove a stronger property named *tameness* [3]. That property on  $\mathbf{V}$  has been used to prove decidability of the membership problem for pseudovarieties obtained from  $\mathbf{V}$  through the application of join, (two-sided) semidirect product, and Mal'cev product with other pseudovarieties.

Almeida and Zeitoun [5] solved the  $\kappa$ -word problem over the pseudovariety  $\mathbf{R}$  of all finite semigroups whose regular  $\mathcal{R}$ -classes are trivial. Their methods have been extended by Moura [9] to the pseudovariety  $\mathbf{DA}$ , consisting of all finite semigroups whose regular  $\mathcal{D}$ -classes are aperiodic subsemigroups. In this paper, we solve the same problem for some of the pseudovarieties of the form  $\mathbf{DRH}$ , containing all finite semigroups whose regular  $\mathcal{R}$ -classes

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lie in a pseudovariety of groups  $H$ . The only condition we impose on  $H$  is quite reasonable: we require that it has a decidable  $\kappa$ -word problem. Further, combining Moura's work with our own, it is expected that the same approach may be extended to  $DO \cap \overline{H}$ , that is, the pseudovariety of all finite semigroups whose regular  $\mathcal{D}$ -classes are orthodox semigroups and whose subgroups lie in  $H$ . The pseudovariety  $DO \cap \overline{H}$  may be considered as a non aperiodic version of  $DA$  in the same way that  $DRH$  may be seen as a non aperiodic version of  $R$ . All of these pseudovarieties are specializations of  $DS$ , the pseudovariety of all finite semigroups whose regular  $\mathcal{D}$ -classes are subsemigroups, which is known to be of particular interest in the Theory of Formal Languages (see, for instance, [11]). Hence, their investigation may lead to a better understanding of  $DS$ .

This paper is organized as follows. Section 2 of preliminaries is divided into four subsections: in the first we set up the general notation; we recall some aspects related with theory of profinite semigroups in the second; we describe the  $\kappa$ -word problem in the third; and we reserve the fourth to the statement of some general facts on the structure of the free pro- $DRH$  semigroup. In Section 3 we introduce  $DRH$ -automata, which are a generalization of  $R$ -automata defined in [5]. We devote Section 4 to the presentation of a canonical form for  $\kappa$ -words over  $DRH$  assuming the knowledge of a canonical form for  $\kappa$ -words over  $H$ . Section 5 is rather technical and serves the purpose of preparing Section 6, in which we describe an algorithm to solve the  $\kappa$ -word problem over  $DRH$ . Finally, in Section 7 we apply our results to the particular case of the pseudovariety  $DRG$ .

## 2. PRELIMINARIES

We assume the reader is familiar with pseudovarieties, (pro)finite semigroups, and the basic topology. For further reading we refer to [1, 2, 12]. Some knowledge of automata theory may be useful, although no use of deep results is made. For this topic, we refer to [10]. A study of pseudovarieties of the form  $DRH$  may be found in [4].

**2.1. Notation.** Given a semigroup  $S$ , we let  $S^I$  represent the monoid obtained by adjoining an identity to  $S$  (even if  $S$  is already a monoid). If  $s_1, \dots, s_n$  are elements in  $S$ , then  $\prod_{i=1}^n s_i$  denotes the product  $s_1 \cdots s_n$ . An infinite sequence  $(s_i)_{i \geq 1} \subseteq S$  defines the *infinite product*  $(\prod_{i=1}^n s_i)_{n \geq 1}$ .

The free semigroup (respectively, monoid) on a (possibly infinite) set  $C$  is denoted  $C^+$  (respectively,  $C^*$ ). Elements in  $C^*$  are called *words*. The *empty word* of  $C^*$  is the identity element  $\varepsilon$ . The *length* of a word  $u \in C^*$  is  $|u| = 0$  if  $u = \varepsilon$ , and  $|u| = n$  if  $u = c_1 \cdots c_n$ , for certain  $c_1, \dots, c_n \in C$ . The free group on  $C$  is denoted  $FG_C$ , and we denote by  $C^{-1}$  the set  $\{c^{-1} : c \in C\}$  disjoint from  $C$ , where  $c^{-1}$  represents the inverse of  $c$  in  $FG_C$ .

We say that a finite set of symbols is an *alphabet*. Generic alphabets are denoted  $A$ , while  $\Sigma = \{0, 1\}$  is a fixed two-element alphabet.

Let  $\mathcal{A} = \langle V, \rightarrow, q, F \rangle$  be a *deterministic automaton* over an alphabet  $A$  (where  $V$  is the set of *states*,  $\rightarrow$  is the *transition function*, and  $\{q\}$  and  $F$  are the sets of *initial* and *terminal states*, respectively). We write transitions in  $\mathcal{A}$  as  $v \xrightarrow{a} v.a$ , for  $v \in V$  and  $a \in A^*$ . Given a state  $v \in V$ , we denote by  $\mathcal{A}_v$  the sub-automaton of  $\mathcal{A}$  *rooted at*  $v$ , that is, the (deterministic) automaton  $\langle v.A^*, \rightarrow|_{v.A^*}, v, F \cap (v.A^*) \rangle$ .

The symbols  $\mathcal{R}$ ,  $\mathcal{H}$ , and  $\mathcal{D}$  denote some of *Green's relations*. We reserve the letter  $\mathbf{H}$  to denote an arbitrary pseudovariety of groups, and DRH stands for the pseudovariety of all finite semigroups whose regular  $\mathcal{R}$ -classes belong to  $\mathbf{H}$ . Other pseudovarieties playing a role in this work are  $\mathfrak{S}$ , the pseudovariety of all finite semigroups;  $\mathbf{G}$ , the pseudovariety of all finite groups;  $\mathbf{R}$ , the pseudovariety of all finite semigroups with trivial  $\mathcal{R}$ -classes;  $\mathbf{DS}$ , the pseudovariety of all finite semigroups whose regular  $\mathcal{D}$ -classes are subsemigroups;  $\mathbf{DO}$ , the pseudovariety of all finite semigroups whose regular  $\mathcal{D}$ -classes are orthodox subsemigroups; and  $\overline{\mathbf{H}}$ , the pseudovariety of all finite semigroups whose subgroups lie in  $\mathbf{H}$ .

**2.2. Profinite semigroups.** Let  $\mathbf{V}$  be a pseudovariety of semigroups. We denote the *free  $A$ -generated pro- $\mathbf{V}$  semigroup* by  $\overline{\Omega}_A \mathbf{V}$ . Elements of  $\overline{\Omega}_A \mathbf{V}$  are called *pseudowords over  $\mathbf{V}$*  (or simply *pseudowords*, when  $\mathbf{V} = \mathfrak{S}$ ). Let  $\iota : A \rightarrow \overline{\Omega}_A \mathbf{V}$  be the generating mapping of  $\overline{\Omega}_A \mathbf{V}$ . We point out that, unless  $\mathbf{V}$  is the trivial pseudovariety,  $\iota$  is injective. For that reason, we often identify the alphabet  $A$  with its image under  $\iota$ . With this assumption, we obtain that the free semigroup  $A^+$  is a subsemigroup of  $\overline{\Omega}_A \mathbf{V}$  and thus, it is coherent to say that  $I \in (\overline{\Omega}_A \mathbf{V})^I$  is the *empty word/pseudoword*. On the other hand, if  $B \subseteq A$ , then we have an injective continuous homomorphism  $\overline{\Omega}_B \mathbf{V} \rightarrow \overline{\Omega}_A \mathbf{V}$ , induced by the inclusion map  $B \rightarrow A$ . So, we consider  $\overline{\Omega}_B \mathbf{V}$  as a subsemigroup of  $\overline{\Omega}_A \mathbf{V}$ . In turn, if  $\mathbf{W}$  is a subpseudovariety of  $\mathbf{V}$ , then we denote by  $\rho_{\mathbf{V}, \mathbf{W}}$  the natural projection from  $\overline{\Omega}_A \mathbf{V}$  onto  $\overline{\Omega}_A \mathbf{W}$ . We shall write  $\rho_{\mathbf{W}}$  when  $\mathbf{V}$  is clear from the context. Whenever the pseudovariety  $\mathbf{Sl}$  of all finite semilattices is contained in  $\mathbf{V}$ , we denote the projection  $\rho_{\mathbf{Sl}} = \rho_{\mathbf{V}, \mathbf{Sl}}$  by  $c$  and call it the *content function*.

Finally, a *pseudoidentity over  $\mathbf{V}$*  (or simply *pseudoidentity*, when  $\mathbf{V} = \mathfrak{S}$ ) is a formal equality  $u = v$ , with  $u, v \in \overline{\Omega}_A \mathbf{V}$ . We say that a pseudoidentity  $u = v$  holds in a pseudovariety  $\mathbf{W} \subseteq \mathbf{V}$  if the interpretations of  $u$  and  $v$  coincide in every semigroup of  $\mathbf{W}$ . If that is the case, then we write  $u =_{\mathbf{W}} v$ .

**2.3. The  $\kappa$ -word problem.** The *canonical implicit signature*, denoted  $\kappa$ , consists of two implicit operations: the *multiplication*  $\cdot$ , and the  $(\omega - 1)$ -*power*  $^{\omega-1}$ . Each of these operations has a natural interpretation over a given profinite semigroup  $S$ : the multiplication sends each pair  $(s_1, s_2)$  to its product  $s_1 s_2$ , and the  $(\omega - 1)$ -power sends each element  $s$  to the limit  $\lim_{n \geq 1} s^{n!-1}$ . We define  $\kappa$ -terms over an alphabet  $A$  inductively as follows:

- the empty word  $I$  and each letter  $a \in A$  are  $\kappa$ -terms;
- if  $u$  and  $v$  are  $\kappa$ -terms, then  $(u \cdot v)$  and  $(u^{\omega-1})$  are also  $\kappa$ -terms.

Of course, each  $\kappa$ -term may naturally be seen as representing an element of the free  $\kappa$ -semigroup  $\Omega_A^\kappa \mathbf{S}$ , and conversely, for each element of  $\Omega_A^\kappa \mathbf{S}$  there is a (usually non-unique)  $\kappa$ -term representing it. We call  $\kappa$ -words the elements of  $\Omega_A^\kappa \mathbf{S}$ .

Let  $\ell$  be an integer. We may generalize the  $(\omega - 1)$ -power by letting  $x^{\omega+\ell} = \lim_{n \geq 1} x^{n!+\ell}$ . Then, for every  $q \geq 1$ , the expressions  $(x^{\omega-1})^q$  and  $x^{\omega-1}x^q$  represent  $\kappa$ -words (by  $u^q$  we mean  $q$  times the product of  $u$ ), and the equalities  $(x^{\omega-1})^q = x^{\omega-q}$  and  $x^{\omega-1}x^{q+1} = x^{\omega+q-1}$  hold in  $\Omega_A^\kappa \mathbf{S}$ . It is usual to consider the extended implicit signature  $\bar{\kappa}$  that contains the multiplication and all  $(\omega + q)$ -powers (for an integer  $q$ ). We define both  $\bar{\kappa}$ -term and  $\bar{\kappa}$ -word in the same fashion as we defined  $\kappa$ -term and  $\kappa$ -word, respectively. Clearly,  $\kappa$ -words are  $\bar{\kappa}$ -words and conversely, but a  $\bar{\kappa}$ -term may not be a  $\kappa$ -term.

Saying that the  $\kappa$ -word problem over a pseudovariety  $\mathbf{V}$  is decidable amounts to say that there exists an algorithm determining whether the interpretation of two given  $\kappa$ -terms coincides in every semigroup of  $\mathbf{V}$ , that is, whether they define the same element of  $\Omega_A^\kappa \mathbf{V}$ . Although our goal is to solve the  $\kappa$ -word problem over  $\mathbf{DRH}$  (under certain reasonable conditions on  $\mathbf{H}$ ), it shall be useful to consider  $\bar{\kappa}$ -terms instead of  $\kappa$ -terms in the intermediate steps.

The implicit signature  $\bar{\kappa}$  enjoys a nice property that we state here for later reference.

**Lemma 2.1** ([5, Lemma 2.2]). *Let  $u$  be a  $\bar{\kappa}$ -term and let  $u = u_\ell a u_r$  be a factorization of  $u$  such that  $c(u) = c(u_\ell) \uplus \{a\}$ . Then,  $u_\ell$  and  $u_r$  are  $\bar{\kappa}$ -terms.*

**2.4. Structure of free pro-DRH semigroups.** We start with a uniqueness result on factorization of pseudowords.

**Proposition 2.2** ([5, Proposition 2.1]). *Let  $x, y, z, t \in \bar{\Omega}_A \mathbf{S}$  and  $a, b \in A$  be such that  $xay = zbt$ . Suppose that  $a \notin c(x)$  and  $b \notin c(z)$ . If either  $c(x) = c(z)$  or  $c(xa) = c(zb)$ , then  $x = z$ ,  $a = b$ , and  $y = t$ .*

This motivates the definition of *left basic factorization* of a pseudoword  $u \in \bar{\Omega}_A \mathbf{S}$ : it is the unique triple  $\text{lbf}(u) = (u_\ell, a, u_r)$  of  $(\bar{\Omega}_A \mathbf{S})^I \times A \times (\bar{\Omega}_A \mathbf{S})^I$  such that  $u = u_\ell a u_r$ ,  $a \notin c(u_\ell)$ , and  $c(u) = c(u_\ell a)$ . The left basic factorization is also well defined over each pseudovariety  $\mathbf{DRH}$ .

**Proposition 2.3** ([4, Proposition 2.3.1]). *Every element  $u \in \bar{\Omega}_A \mathbf{DRH}$  admits a unique factorization of the form  $u = u_\ell a u_r$  such that  $a \notin c(u_\ell)$  and  $c(u_\ell a) = c(u)$ .*

Then, whenever  $u \in \bar{\Omega}_A \mathbf{DRH}$ , we also say that the triple  $\text{lbf}(u) = (u_\ell, a, u_r)$  described in Proposition 2.3 is the left basic factorization of  $u$ .

We may iterate the left basic factorization of a pseudoword  $u$  (over  $\mathbf{DRH}$ ) as follows. Set  $u'_0 = u$ . For  $k \geq 0$ , if  $u'_k \neq I$ , then we let  $(u_{k+1}, a_{k+1}, u'_{k+1})$  be the left basic factorization of  $u'_k$ . Since the contents  $(c(u_k a_k))_{k \geq 1}$  form a decreasing sequence for inclusion, either there exists an index  $k$  such that  $u'_k = I$  or, for all  $m \geq k$ ,  $c(u_k a_k) = c(u_m a_m)$ . The *cumulative content* of  $u$  is  $\bar{c}(u) = \emptyset$  in the former case, and it is  $\bar{c}(u) = c(u_k a_k)$  otherwise.

In particular, Proposition 2.3 yields that the cumulative content of a pseudoword is completely determined by its projection onto  $\overline{\Omega}_A\mathcal{R}$ . We denote the factor  $u_k a_k$  by  $\text{lbf}_k(u)$ , whenever it is defined and we write  $\text{lbf}_\infty(u) = (u_1 a_1, \dots, u_k a_k, I, I, \dots)$  if  $u'_k = I$ , and  $\text{lbf}_\infty(u) = (u_k a_k)_{k \geq 1}$  otherwise. We further define the *irregular* and *regular* parts of  $u$ , respectively denoted  $\text{irr}(u)$  and  $\text{reg}(u)$ : if  $\vec{c}(u) = \emptyset$ , then  $\text{irr}(u) = u$  and  $\text{reg}(u) = I$ ; if  $\vec{c}(u) = c(u'_k)$  and  $k$  is minimal for this equality, then  $\text{irr}(u) = \text{lbf}_1(u) \cdots \text{lbf}_k(u)$  and  $\text{reg}(u) = u'_k$ . This terminology is explained by the following result.

**Proposition 2.4** ([4, Corollary 6.1.5]). *Let  $u \in \overline{\Omega}_A\text{DRH}$ . Then,  $u$  is regular if and only if  $c(u) = \vec{c}(u)$  (and, hence,  $\text{reg}(u) = u$ ).*

Suppose that  $\vec{c}(u) \neq \emptyset$ . Since  $\overline{\Omega}_A\mathcal{S}$  is a compact monoid, it follows that the infinite product  $(\text{lbf}_1(u) \cdots \text{lbf}_k(u))_{k \geq 1}$  has an accumulation point, and it is not hard to see that any two of its accumulation points are  $\mathcal{R}$ -equivalent. Furthermore, if all the factors  $\text{lbf}_k(u)$  have the same content, then the  $\mathcal{R}$ -class in which the accumulation points lie is regular [4, Proposition 2.1.4]. On the other hand, the regular  $\mathcal{R}$ -classes of  $\overline{\Omega}_A\text{DRH}$  are groups. Hence, in this case, we may define the *idempotent designated* by the infinite product  $(\text{lbf}_1(u) \cdots \text{lbf}_k(u))_{k \geq 1}$  to be the identity of the group to which its accumulation points belong.

Together with Lemma 2.1, the next result is behind the properties of  $\overline{\Omega}_A\text{DRH}$  that we use most often in the sequel.

**Proposition 2.5** ([4, Proposition 5.1.2]). *Let  $\mathcal{V}$  be a pseudovariety such that the inclusions  $\mathcal{H} \subseteq \mathcal{V} \subseteq \text{DO} \cap \overline{\mathcal{H}}$  hold. If  $e$  is an idempotent of  $\overline{\Omega}_A\mathcal{V}$  and if  $H_e$  is its  $\mathcal{H}$ -class, then letting  $\psi_e(a) = eae$  for each  $a \in c(e)$  defines a unique homeomorphism  $\psi_e : \overline{\Omega}_{c(e)}\mathcal{H} \rightarrow H_e$  whose inverse is the restriction of  $\rho_{\mathcal{V}, \mathcal{H}}$  to  $H_e$ .*

The following consequence is not hard to derive.

**Corollary 2.6.** *Let  $u$  be a pseudoword and  $v, w \in (\overline{\Omega}_A\mathcal{S})^I$  be such that  $c(v) \cup c(w) \subseteq \vec{c}(u)$  and  $v =_{\mathcal{H}} w$ . Then, the pseudovariety  $\text{DRH}$  satisfies  $uv = uw$ .*

We proceed with the statement of two known facts about  $\text{DRH}$ . Their proofs may be found in [7].

**Lemma 2.7.** *Let  $u, v$  be pseudowords. Then,  $\rho_{\text{DRH}}(u)$  and  $\rho_{\text{DRH}}(v)$  lie in the same  $\mathcal{R}$ -class if and only if the pseudovariety  $\text{DRH}$  satisfies  $\text{lbf}_\infty(u) = \text{lbf}_\infty(v)$ .*

**Lemma 2.8.** *Let  $u, v \in \overline{\Omega}_A\mathcal{S}$  and  $u_0, v_0 \in (\overline{\Omega}_A\mathcal{S})^I$  be such that  $c(u_0) \subseteq \vec{c}(u)$  and  $c(v_0) \subseteq \vec{c}(v)$ . Then, the pseudovariety  $\text{DRH}$  satisfies  $uu_0 = vv_0$  if and only if it satisfies  $u \mathcal{R} v$  and if, in addition, the pseudovariety  $\mathcal{H}$  satisfies  $uu_0 = vv_0$ . In particular, by taking  $u_0 = I = v_0$ , we get that  $u =_{\text{DRH}} v$  if and only if  $u \mathcal{R} v$  modulo  $\text{DRH}$  and  $u =_{\mathcal{H}} v$ .*

## 3. DRH-AUTOMATA

We start by introducing the notion of a DRH-automaton.

**Definition 3.1.** *An  $A$ -labeled DRH-automaton is a tuple*

$$\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_{\mathbf{H}}, \lambda \rangle,$$

where  $\langle V, \rightarrow, \mathbf{q}, F \rangle$  is a nonempty deterministic trim automaton over  $\Sigma$  and  $\lambda_{\mathbf{H}} : V \rightarrow (\overline{\Omega}_A \mathbf{H})^I$  and  $\lambda : V \rightarrow A \uplus \{\varepsilon\}$  are functions. We further require that  $\mathcal{A}$  satisfies the following conditions (A.1)–(A.6).

- (A.1) the set of final states is  $F = \lambda^{-1}(\varepsilon)$  and  $\lambda_{\mathbf{H}}(F) = \{I\}$ ;
- (A.2) there is no outgoing transition from  $F$ ;
- (A.3) for every  $\mathbf{v} \in V \setminus F$ , both  $\mathbf{v}.0$  and  $\mathbf{v}.1$  are defined;
- (A.4) for every  $\mathbf{v} \in V \setminus F$ , the equality  $\lambda(\mathbf{v}.\Sigma^*) = \lambda(\mathbf{v}.0\Sigma^*) \uplus \{\lambda(\mathbf{v})\}$  holds.

We observe that if conditions (A.1)–(A.4) hold for  $\mathcal{A}$ , then the reduct  $\mathcal{A}_{\mathbf{R}} = \langle V, \rightarrow, \mathbf{q}, F, \lambda \rangle$  is an  $A$ -labeled  $\mathbf{R}$ -automaton (see [5, Definition 3.11]). Since the cumulative content of a pseudoword over DRH depends only on its projection onto  $\overline{\Omega}_A \mathbf{R}$ , and hence, also its regularity, we may use the known results for the word problem in  $\mathbf{R}$  (namely, [5, Theorem 3.21]) as intuition for defining the length  $\|\mathcal{A}\|$ , the regularity index  $\mathbf{r.ind}(\mathcal{A})$  and the cumulative content  $\tilde{c}(\mathcal{A})$  of a DRH-automaton  $\mathcal{A}$  from the knowledge of its reduct  $\mathcal{A}_{\mathbf{R}}$ . We set:

$$\begin{aligned} \|\mathcal{A}\| &= \sup\{k \geq 0 : \mathbf{q}.1^k \text{ is defined}\}; \\ \mathbf{r.ind}(\mathcal{A}) &= \begin{cases} \infty, & \text{if } \|\mathcal{A}\| < \infty; \\ \min\{m \geq 0 : \forall k \geq m \quad \lambda(\mathbf{q}.1^k \Sigma^*) = \lambda(\mathbf{q}.1^m \Sigma^*)\}, & \text{otherwise;} \end{cases} \\ \tilde{c}(\mathcal{A}) &= \begin{cases} \emptyset, & \text{if } \|\mathcal{A}\| < \infty; \\ \lambda(\mathbf{q}.1^{\mathbf{r.ind}(\mathcal{A})} \Sigma^*), & \text{otherwise.} \end{cases} \end{aligned}$$

We are now able to state the further required properties for  $\mathcal{A}$ :

- (A.5) if  $\mathbf{v} \in V \setminus F$ , then  $\lambda_{\mathbf{H}}(\mathbf{v}) = I$  if and only if  $\|\mathcal{A}_{\mathbf{v}.0}\| < \infty$ ;
- (A.6) if  $\mathbf{v} \in V \setminus F$  and  $\|\mathcal{A}_{\mathbf{v}.0}\| = \infty$ , then  $\lambda_{\mathbf{H}}(\mathbf{v}) \in \overline{\Omega}_{\tilde{c}(\mathcal{A}_{\mathbf{v}.0})} \mathbf{H}$ .

We say that  $\mathcal{A}$  is a DRH-tree if it is a DRH-automaton such that for every  $\mathbf{v} \in V$  there exists a unique  $\alpha \in \Sigma^*$  such that  $\mathbf{q}.\alpha = \mathbf{v}$ .

**Definition 3.2.** *We say that two DRH-automata  $\mathcal{A}_i = \langle V_i, \rightarrow_i, \mathbf{q}_i, F_i, \lambda_{i,\mathbf{H}}, \lambda_i \rangle$ ,  $i = 1, 2$ , are isomorphic if there exists a bijection  $f : V_1 \rightarrow V_2$  such that*

- $f(\mathbf{q}_1) = \mathbf{q}_2$ ;
- for every  $\mathbf{v} \in V_1$  and  $\alpha \in \Sigma$ ,  $f(\mathbf{v}) \cdot \alpha = f(\mathbf{v} \cdot \alpha)$ ;
- for every  $\mathbf{v} \in V_1$ , the equalities  $\lambda_{1,\mathbf{H}}(\mathbf{v}) = \lambda_{2,\mathbf{H}}(f(\mathbf{v}))$  and  $\lambda_1(\mathbf{v}) = \lambda_2(f(\mathbf{v}))$  hold.

Isomorphic DRH-automata are essentially the same, up to the name of the states. Therefore, we consider DRH-automata only up to isomorphism.

We denote the trivial DRH-automaton by  $\mathbf{1}$  and the set of all  $A$ -labeled DRH-automata by  $\mathbb{A}_A$ .

**Definition 3.3.** Let  $k \geq 0$  and  $\mathcal{A}_i = \langle V_i, \rightarrow_i, \mathbf{q}_i, F_i, \lambda_{i,H}, \lambda_i \rangle$ ,  $i = 1, 2$ , be two DRH-automata. We say that  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $k$ -equivalent if

$$(1) \quad \forall \alpha \in \Sigma^*, |\alpha| \leq k \implies \begin{cases} \lambda_1(\mathbf{q}_1.\alpha) = \lambda_2(\mathbf{q}_2.\alpha); \\ \lambda_{1,H}(\mathbf{q}_1.\alpha) = \lambda_{2,H}(\mathbf{q}_2.\alpha). \end{cases}$$

If  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $k$ -equivalent for every  $k \geq 0$ , then we say that they are equivalent. We write  $\mathcal{A}_1 \sim_k \mathcal{A}_2$  (respectively,  $\mathcal{A}_1 \sim \mathcal{A}_2$ ), when  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $k$ -equivalent (respectively, equivalent). We further agree that (1) means that either both equalities hold or both  $\mathbf{q}_1.\alpha$  and  $\mathbf{q}_2.\alpha$  are undefined.

Observe that equivalent DRH-trees are necessarily isomorphic.

The following lemma is useful when defining a bijective correspondence between the equivalence classes of  $\mathbb{A}_A$  and the  $\mathcal{R}$ -classes of  $\overline{\Omega}_A \text{DRH}$ . Although its proof is analogous to the proof of [5, Lemma 3.16], we include it for the sake of completeness.

**Lemma 3.4.** Every DRH-automaton has a unique equivalent DRH-tree.

*Proof.* Take a DRH-automaton  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$  and let  $\mathcal{T} = \langle V', \rightarrow', \mathbf{q}', F', \lambda'_H, \lambda' \rangle$  be the DRH-tree defined as follows. We set  $V' = \{\alpha \in \Sigma^* : \mathbf{q}.\alpha \text{ is defined}\}$  and put  $\mathbf{q}' = \varepsilon$ . The labels of each state  $\alpha \in V'$  are given by  $\lambda'_H(\alpha) = \lambda_H(\mathbf{q}.\alpha)$  and by  $\lambda'(\alpha) = \lambda(\mathbf{q}.\alpha)$ . We also take  $F' = \lambda'^{-1}(\varepsilon)$ . Finally, the transitions in  $\mathcal{T}$  are given by  $\alpha.0 = \alpha 0$  and by  $\alpha.1 = \alpha 1$ , whenever  $\lambda'(\alpha) \neq \varepsilon$ . It is a routine matter to check that  $\mathcal{T}$  is a DRH-tree equivalent to  $\mathcal{A}$ .  $\square$

Given a DRH-automaton  $\mathcal{A}$ , we denote by  $\vec{\mathcal{A}} = \langle \vec{V}, \rightarrow, \vec{\mathbf{q}}, \vec{F}, \vec{\lambda}_H, \vec{\lambda} \rangle$  the unique DRH-tree which is equivalent to  $\mathcal{A}$ . Denoting both transition functions of  $\mathcal{A}$  and of  $\vec{\mathcal{A}}$  by  $\rightarrow$  is an abuse of notation justified by the construction made in the proof of Lemma 3.4. Given  $0 \leq i \leq \|\mathcal{A}\|$ , we denote by  $\mathcal{A}_{[i]}$  the DRH-subtree rooted at  $\vec{\mathbf{q}}.1^i 0$ .

**Notation 3.5.** Let  $u \in \overline{\Omega}_A \text{DRH}$  and  $v \in \overline{\Omega}_A H$  be such that  $c(v) \subseteq \vec{c}(u)$ . By Corollary 2.6, the set  $up_{\text{DRH},H}^{-1}(v)$  is a singleton. It is convenient to denote by  $uv$  the unique element of  $up_{\text{DRH},H}^{-1}(v)$ . In this case, the notation  $\rho_H(uv)$  refers to the element  $\rho_H(uv) = \rho_H(u) v$  of  $\overline{\Omega}_A H$ .

**Definition 3.6.** Let  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$  be an  $A$ -labeled DRH-automaton. The value  $\pi(\mathcal{A})$  of  $\mathcal{A}$  in  $(\overline{\Omega}_A \text{DRH})^I$  is inductively defined as follows:

- if  $\mathcal{A} = \mathbf{1}$ , then  $\pi(\mathcal{A}) = I$ ;
- otherwise, we consider two different cases according to whether or not  $\|\mathcal{A}\| < \infty$ .
  - If  $\|\mathcal{A}\| < \infty$ , then we set

$$\pi(\mathcal{A}) = \prod_{i=0}^{\|\mathcal{A}\|-1} \pi(\mathcal{A}_{[i]}) \lambda_H(\mathbf{q}.1^i) \lambda(\mathbf{q}.1^i).$$

- If  $\|\mathcal{A}\| = \infty$ , then we first define the idempotent associated to  $\mathcal{A}$ , denoted  $\text{id}(\mathcal{A})$ . Noticing that, for  $k \geq \text{r.ind}(\mathcal{A})$ , all the elements  $\pi(\mathcal{A}_{[k]})\lambda_{\mathbf{H}}(\mathbf{q}.1^k)\lambda(\mathbf{q}.1^k)$  have the same content, we let  $\text{id}(\mathcal{A})$  be the idempotent designated by the infinite product

$$(2) \quad (\pi(\mathcal{A}_{[\text{r.ind}(\mathcal{A})]})\lambda_{\mathbf{H}}(\mathbf{q}.1^{\text{r.ind}(\mathcal{A})})\lambda(\mathbf{q}.1^{\text{r.ind}(\mathcal{A})}) \cdots \pi(\mathcal{A}_{[k]})\lambda_{\mathbf{H}}(\mathbf{q}.1^k)\lambda(\mathbf{q}.1^k))_{k \geq \text{r.ind}(\mathcal{A})}.$$

Then, we take

$$\pi(\mathcal{A}) = \left( \prod_{i=0}^{\text{r.ind}(\mathcal{A})-1} \pi(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i)\lambda(\mathbf{q}.1^i) \right) \cdot \text{id}(\mathcal{A}).$$

We also define the value of the irregular part of  $\mathcal{A}$ :

$$\pi_{\text{irr}}(\mathcal{A}) = \prod_{i=0}^{\min\{\|\mathcal{A}\|, \text{r.ind}(\mathcal{A})\}-1} \pi(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i)\lambda(\mathbf{q}.1^i).$$

If  $\|\mathcal{A}\| < \infty$ , then we set  $\text{id}(\mathcal{A}) = I$ . Using this notation, we have the equality

$$(3) \quad \pi(\mathcal{A}) = \pi_{\text{irr}}(\mathcal{A}) \cdot \text{id}(\mathcal{A}).$$

The next result is a simple observation that we state for later reference.

**Lemma 3.7.** *Given a DRH-automaton  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_{\mathbf{H}}, \lambda \rangle$ , the following equalities hold:*

$$\begin{aligned} \text{lbf}_{i+1}(\pi(\mathcal{A})) &= \pi(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i)\lambda(\mathbf{q}.1^i), \text{ whenever } \text{lbf}_{i+1}(\pi(\mathcal{A})) \text{ is defined;} \\ \text{irr}(\pi(\mathcal{A})) &= \pi_{\text{irr}}(\mathcal{A}); \\ \vec{c}(\mathcal{A}) &= \vec{c}(\pi(\mathcal{A})). \end{aligned}$$

In particular, for a certain  $u \in \overline{\Omega}_A \text{DRH}$ , the elements  $\pi(\mathcal{A})$  and  $u$  are  $\mathcal{R}$ -equivalent if and only if  $\pi_{\text{irr}}(\mathcal{A}) = \text{irr}(u)$  and  $\text{id}(\mathcal{A}) \mathcal{R} \text{reg}(u)$ .  $\square$

Since the value of a DRH-automaton  $\mathcal{A}$  depends only on the unique DRH-tree  $\vec{\mathcal{A}}$  lying in the  $\sim$ -class of  $\mathcal{A}$ , there is a well defined map  $\bar{\pi} : \mathbb{A}_A / \sim \rightarrow (\overline{\Omega}_A \text{DRH})^I / \mathcal{R}$  which sends a class  $\mathcal{A} / \sim$  to the  $\mathcal{R}$ -class of the value of  $\vec{\mathcal{A}}$ . This map is, in effect, a bijection.

**Theorem 3.8.** *The map  $\bar{\pi}$  is bijective.*

*Proof.* To prove that  $\bar{\pi}$  is injective, we consider two DRH-automata  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_{\mathbf{H}}, \lambda \rangle$  and  $\mathcal{A}' = \langle V', \rightarrow', \mathbf{q}', F', \lambda'_{\mathbf{H}}, \lambda' \rangle$  such that  $\pi(\mathcal{A}) \mathcal{R} \pi(\mathcal{A}')$  and we argue by induction on  $|c(\pi(\mathcal{A}))| = |c(\pi(\mathcal{A}'))|$ .

If  $|c(\pi(\mathcal{A}))| = 0$ , then  $\mathcal{A} = \mathbf{1} = \mathcal{A}'$  and there is nothing to prove.

Suppose that  $|c(\pi(\mathcal{A}))| > 0$ . We claim that  $\mathcal{A}_{[i]} = \mathcal{A}'_{[i]}$  for all  $0 \leq i \leq \|\mathcal{A}\| - 1$ . Indeed, by Lemma 2.7, the values  $\pi(\mathcal{A})$  and  $\pi(\mathcal{A}')$  lie in the same  $\mathcal{R}$ -class if and only if  $\text{lbf}_{\infty}(\pi(\mathcal{A})) = \text{lbf}_{\infty}(\pi(\mathcal{A}'))$ . But, by Lemma 3.7, the



equalities

$$\begin{aligned}\text{lbf}_{i+1}(\pi(\mathcal{A})) &= \pi(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i)\lambda(\mathbf{q}.1^i) \\ \text{lbf}_{i+1}(\pi(\mathcal{A}')) &= \pi(\mathcal{A}'_{[i]})\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i)\lambda'(\mathbf{q}'.1^i)\end{aligned}$$

hold, whenever the first members are defined. Hence, we get the following:

$$\begin{aligned}(4) \quad & \|\mathcal{A}\| = \|\mathcal{A}'\|, \\ & \pi(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i) = \pi(\mathcal{A}'_{[i]})\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i), \text{ for } 0 \leq i \leq \|\mathcal{A}\| - 1, \\ & \lambda(\mathbf{q}.1^i) = \lambda'(\mathbf{q}'.1^i), \text{ for } 0 \leq i \leq \|\mathcal{A}\| - 1.\end{aligned}$$

Since, by (A.6), the inclusions  $c(\lambda_{\mathbf{H}}(\mathbf{q}.1^i)) \subseteq \vec{c}(\pi(\mathcal{A}_{[i]}))$  and  $c(\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i)) \subseteq \vec{c}(\pi(\mathcal{A}'_{[i]}))$  hold, we also have  $\pi(\mathcal{A}_{[i]}) \mathcal{R} \pi(\mathcal{A}'_{[i]})$ . By induction hypothesis, that implies  $\mathcal{A}_{[i]} = \mathcal{A}'_{[i]}$  (recall that  $\mathcal{A}_{[i]}$  and  $\mathcal{A}'_{[i]}$  are both DRH-trees, and each equivalence class has a unique DRH-tree).

To conclude that  $\bar{\pi}$  is injective, it remains to show that, for  $0 \leq i \leq \|\mathcal{A}\| - 1$ , the labels  $\lambda_{\mathbf{H}}(\mathbf{q}.1^i)$  and  $\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i)$  coincide. When  $\vec{c}(\mathcal{A}_{[i]}) = \emptyset = \vec{c}(\mathcal{A}'_{[i]})$ , Property (A.6) guarantees that  $\lambda_{\mathbf{H}}(\mathbf{q}.1^i) = I = \lambda'_{\mathbf{H}}(\mathbf{q}'.1^i)$ . Otherwise, we have

$$\begin{aligned}\pi_{\text{irr}}(\mathcal{A}_{[i]})\text{id}(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i) &= \pi(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i) \stackrel{(4)}{=} \pi(\mathcal{A}'_{[i]})\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i) \\ &= \pi_{\text{irr}}(\mathcal{A}'_{[i]})\text{id}(\mathcal{A}'_{[i]})\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i),\end{aligned}$$

which in turn implies that

$$\text{id}(\mathcal{A}_{[i]})\lambda_{\mathbf{H}}(\mathbf{q}.1^i) = \text{id}(\mathcal{A}'_{[i]})\lambda'_{\mathbf{H}}(\mathbf{q}'.1^i).$$

Since  $\rho_{\mathbf{H}}(\text{id}(\mathcal{A}_{[i]}))$  and  $\rho_{\mathbf{H}}(\text{id}(\mathcal{A}'_{[i]}))$  are both the identity of  $\bar{\Omega}_A \mathbf{H}$  we obtain the equality  $\lambda_{\mathbf{H}}(\mathbf{q}.1^i) = \lambda'_{\mathbf{H}}(\mathbf{q}'.1^i)$ .

Let us prove that  $\bar{\pi}$  is surjective. We proceed again by induction, this time on  $|c(w)|$ , for  $w \in (\bar{\Omega}_A \text{DRH})^I$ .

If  $c(w)$  is the empty set, then we have  $[w]_{\mathcal{R}} = \{I\} = \{\pi(\mathbf{1})\} = \bar{\pi}(\mathbf{1}/\sim)$ .

If  $w \neq I$ , then we let  $w = w_0 a_0 \cdots w_k a_k w'_k$  be the  $k$ -th iteration of the left basic factorization of  $w$  (whenever it is defined). For each  $0 \leq i \leq \lceil w \rceil - 1$ , we have  $c(w_i) \subsetneq c(w)$  and so, by induction hypothesis, there exists a DRH-tree  $\mathcal{A}_i = \langle V_i, \rightarrow_i, \mathbf{q}_i, F_i, \lambda_{i,\mathbf{H}}, \lambda_i \rangle$  such that  $\pi(\mathcal{A}_i) \mathcal{R} w_i$ . In particular, the equality  $\pi_{\text{irr}}(\mathcal{A}_i) = \text{irr}(w_i)$  holds and consequently,  $\mathbf{H}$  satisfies

$$(5) \quad \pi(\mathcal{A}_i) \cdot \text{reg}(w_i) = \pi_{\text{irr}}(\mathcal{A}_i) \cdot \text{id}(\mathcal{A}_i) \cdot \text{reg}(w_i) = \text{irr}(w_i) \cdot 1 \cdot \text{reg}(w_i) = w_i.$$

On the other hand, since  $c(\text{reg}(w_i)) = \vec{c}(\text{id}(\mathcal{A}_i))$ , we deduce that  $\text{id}(\mathcal{A}_i) \cdot \text{reg}(w_i)$  is  $\mathcal{R}$ -equivalent to  $\text{id}(\mathcal{A}_i)$ . Consequently, the pseudowords  $w_i$  and  $\pi(\mathcal{A}_i) \cdot \text{reg}(w_i)$  are  $\mathcal{R}$ -equivalent as well. This relation together with (5) imply, by Lemma 2.8, that the equality  $\pi(\mathcal{A}_i) \cdot \text{reg}(w_i) = w_i$  holds.

Now, we construct a DRH-tree  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_{\mathbf{H}}, \lambda \rangle$  as follows:

$$\bullet \quad V = \begin{cases} \{\mathbf{v} \in V_i : i \geq 0\} \uplus \{\mathbf{v}_i\}_{i \geq 0}, & \text{if } \lceil w \rceil = \infty; \\ \{\mathbf{v} \in V_i : i = 0, \dots, \lceil w \rceil - 1\} \uplus \{\mathbf{v}_i\}_{i=0}^{\lceil w \rceil - 1} \uplus \{\mathbf{v}_\varepsilon\}, & \text{if } \lceil w \rceil < \infty; \end{cases}$$

- $\mathbf{q} = \mathbf{v}_0$ ;
- $F = \begin{cases} \{\mathbf{v} \in F_i : i \geq 0\}, & \text{if } \lceil w \rceil = \infty; \\ \{\mathbf{v} \in F_i : i = 0, \dots, \lceil w \rceil - 1\} \uplus \{\mathbf{v}_\varepsilon\}, & \text{if } \lceil w \rceil < \infty; \end{cases}$
- $\lambda_{\mathbf{H}}(\mathbf{v}_i) = \rho_{\mathbf{H}}(\text{reg}(w_i))$  and  $\lambda(\mathbf{v}_i) = a_i$  for  $i = 0, \dots, \lceil w \rceil - 1$ ;
- $\lambda(\mathbf{v}_\varepsilon) = \varepsilon$ , if  $\lceil w \rceil$  is finite;
- $\mathbf{v}_i.0 = \mathbf{q}_i$  and  $\mathbf{v}_i.1 = \begin{cases} \mathbf{v}_{i+1}, & \text{if } i < \lceil w \rceil - 1; \\ \mathbf{v}_\varepsilon, & \text{if } i = \lceil w \rceil - 1; \end{cases}$
- transitions and labelings on  $V_i$  are given by those of  $\mathcal{A}_i$ .

Then it is easy to check that  $\mathcal{A}$  is a DRH-tree and that  $\bar{\pi}(\mathcal{A}/\sim) = \lceil w \rceil_{\mathcal{R}}$ .  $\square$

Suppose that we are given two DRH-automata  $\mathcal{A}_i = \langle V_i, \rightarrow_i, \mathbf{q}_i, F_i, \lambda_{i,\mathbf{H}}, \lambda_i \rangle$ ,  $i = 0, 1$ , a letter  $a \in A$  such that  $\lambda(V_1) \subseteq \lambda(V_0) \uplus \{a\}$  and a pseudoword  $u$  such that  $c(u) \subseteq \bar{c}(\mathcal{A}_0)$ . Then, we denote by  $(\mathcal{A}_0, u \mid a, \mathcal{A}_1)$  the DRH-automaton  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_{\mathbf{H}}, \lambda \rangle$ , where

- $V = V_0 \uplus V_1 \uplus \{\mathbf{q}\}$ ;
- $\mathbf{q}.0 = \mathbf{q}_0$  and  $\mathbf{q}.1 = \mathbf{q}_1$ ;
- $F = F_0 \uplus F_1$ ;
- $\lambda_{\mathbf{H}}(\mathbf{q}) = \rho_{\mathbf{H}}(u)$  and  $\lambda(\mathbf{q}) = a$ ;
- all the other transitions and labels are given by those of  $\mathcal{A}_0$  and  $\mathcal{A}_1$ .

Given an element  $w$  of  $(\bar{\Omega}_A \mathbf{S})^I$ , we denote by  $\mathcal{T}(w)$  the DRH-tree representing the  $\sim$ -class  $\bar{\pi}^{-1}([\rho_{\text{DRH}}(w)]_{\mathcal{R}})$ . With a little abuse of notation, when  $w \in (\bar{\Omega}_A \text{DRH})^I$ , we use  $\mathcal{T}(w)$  to denote the unique DRH-tree in the  $\sim$ -class  $\bar{\pi}^{-1}(\lceil w \rceil_{\mathcal{R}})$ . Later, we shall see that, for every  $\kappa$ -word  $w$ , there exists a finite DRH-automaton  $\mathcal{A}$  in the  $\sim$ -class of  $\mathcal{T}(w)$  (Corollary 4.6).

**Lemma 3.9.** *Let  $w$  be a pseudoword and write  $\text{lbf}(w) = (w_\ell, a, w_r)$ . Then, we have the equality  $\mathcal{T}(w) = (\mathcal{T}(w_\ell), \text{reg}(w_\ell) \mid a, \mathcal{T}(w_r))$ .*

*Proof.* Write

$$\begin{aligned} \mathcal{T}' &= (\mathcal{T}(w_\ell), \text{reg}(w_\ell) \mid a, \mathcal{T}(w_r)) = \langle V, \rightarrow, \mathbf{q}, F, \lambda_{\mathbf{H}}, \lambda \rangle; \\ \mathcal{T}(w_\ell) &= \langle V_0, \rightarrow_0, \mathbf{q}_0, F_0, \lambda_{0,\mathbf{H}}, \lambda_0 \rangle; \\ \mathcal{T}(w_r) &= \langle V_1, \rightarrow_1, \mathbf{q}_1, F_1, \lambda_{1,\mathbf{H}}, \lambda_1 \rangle. \end{aligned}$$

The claim amounts to proving that  $\pi(\mathcal{T}') \mathcal{R} w$  modulo DRH. By definition of  $\mathcal{T}'$ , we have  $\|\mathcal{T}'\| < \infty$  if and only if  $\|\mathcal{T}(w_r)\| < \infty$ . We start by proving that  $\pi(\mathcal{T}')$  and  $\pi(\mathcal{T}(w_\ell))\rho_{\mathbf{H}}(\text{reg}(w_\ell))a \cdot \pi(\mathcal{T}(w_r))$  belong to the same  $\mathcal{R}$ -class. It is worth noticing that, for every  $1 \leq i \leq \|\mathcal{T}'\|$ , we have the following equality:

$$(6) \quad \mathcal{T}'_{[i]} = \mathcal{T}'_{\mathbf{q}.1^i 0} = \mathcal{T}(w_r)_{\mathbf{q}_1.1^{i-1} 0} = \mathcal{T}(w_r)_{[i-1]}.$$

First, assume that  $\|\mathcal{T}'\| < \infty$ . Then, we have  $\|\mathcal{T}'\| = \|\mathcal{T}(w_r)\| + 1$ . Following Definition 3.6 and the construction of  $\mathcal{T}'$ , we may compute

$$\begin{aligned}
 \pi(\mathcal{T}') &= \prod_{i=0}^{\|\mathcal{T}(w_r)\|} \pi(\mathcal{T}'_{[i]}) \lambda_H(\mathbf{q}.1^i) \lambda(\mathbf{q}.1^i) \\
 &= \pi(\mathcal{T}'_{\mathbf{q}.0}) \lambda_H(\mathbf{q}) \lambda(\mathbf{q}) \cdot \prod_{i=0}^{\|\mathcal{T}(w_r)\|-1} \pi(\mathcal{T}'_{[i+1]}) \lambda_H(\mathbf{q}.1^{i+1}) \lambda(\mathbf{q}.1^{i+1}) \\
 (7) \quad &\stackrel{(6)}{=} \pi(\mathcal{T}(w_\ell)) \rho_H(\text{reg}(w_\ell)) a \cdot \pi(\mathcal{T}(w_r)).
 \end{aligned}$$

Now, we suppose that  $\|\mathcal{T}'\| = \infty$ . In that case,  $\text{r.ind}(\mathcal{T}')$  is either  $\text{r.ind}(\mathcal{T}(w_r))$  or  $\text{r.ind}(\mathcal{T}(w_r)) + 1$  according to whether  $\rho_{\text{DRH}}(w)$  is regular (in which case, it is 0) or not, respectively. Suppose that  $\rho_{\text{DRH}}(w)$  is not regular. We compute

$$\begin{aligned}
 \pi(\mathcal{T}') &= \prod_{i=0}^{\text{r.ind}(\mathcal{T}(w_r))} \pi(\mathcal{T}'_{[i]}) \lambda_H(\mathbf{q}.1^i) \lambda(\mathbf{q}.1^i) \cdot \text{id}(\mathcal{T}') \\
 &= \pi(\mathcal{T}'_{\mathbf{q}.0}) \lambda_H(\mathbf{q}) \lambda(\mathbf{q}) \cdot \left( \prod_{i=0}^{\text{r.ind}(\mathcal{T}(w_r))-1} \pi(\mathcal{T}'_{[i+1]}) \lambda_H(\mathbf{q}.1^{i+1}) \lambda(\mathbf{q}.1^{i+1}) \right) \cdot \text{id}(\mathcal{T}') \\
 &\stackrel{(6)}{=} \pi(\mathcal{T}(w_\ell)) \rho_H(\text{reg}(w_\ell)) a \cdot \pi_{\text{irr}}(\mathcal{T}(w_r)) \cdot \text{id}(\mathcal{T}').
 \end{aligned}$$

Now,  $\text{id}(\mathcal{T}')$  is the idempotent designated by the infinite product

$$(\pi(\mathcal{T}'_{[r.\text{ind}(\mathcal{T}')]}) \lambda_H(\mathbf{q}.1^{r.\text{ind}(\mathcal{T}')})) \lambda(\mathbf{q}.1^{r.\text{ind}(\mathcal{T}')})) \cdots \pi(\mathcal{T}'_k) \lambda_H(\mathbf{q}.1^k) \lambda(\mathbf{q}.1^k))_{k \geq r.\text{ind}(\mathcal{T}')}.$$

Hence, by (6), we have  $\text{id}(\mathcal{T}') = \text{id}(\mathcal{T}(w_r))$ , and so, the equality (7) yields

$$(8) \quad \pi(\mathcal{T}') \mathcal{R} \pi(\mathcal{T}(w_\ell)) \rho_H(w_\ell) a \cdot \pi(\mathcal{T}(w_r)).$$

Now, we need to establish the equality  $w_\ell = \pi(\mathcal{T}(w_\ell)) \rho_H(\text{reg}(w_\ell))$ . But, using Lemma 2.8, that is immediate, since  $w_\ell \mathcal{R} \pi(\mathcal{T}(w_\ell)) \text{reg}(\overline{w}_\ell)$  modulo DRH and, by Lemma 3.7,  $\mathbf{H}$  satisfies  $\pi(\mathcal{T}(w_\ell)) \rho_H(\text{reg}(w_\ell)) = \text{irr}(w_\ell) \cdot \text{id}(\mathcal{T}(w_\ell)) \cdot \text{reg}(w_\ell) = w_\ell$ . Hence, it follows from (8) that  $w = w_\ell \cdot a \cdot w_r \mathcal{R} \pi(\mathcal{T}')$ , as intended.

The case where  $\rho_{\text{DRH}}(w)$  is regular is handled similarly.  $\square$

The value of a path  $\mathbf{q}_0 \xrightarrow{\alpha_0} \mathbf{q}_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathbf{q}_{n+1}$  in a DRH-automaton  $\mathcal{A}$  is given by  $\prod_{i=0}^n (\alpha_i, \lambda_{\mathbf{H}, \alpha_i}(\mathbf{q}_i), \lambda(\mathbf{q}_i)) \in (\Sigma \times (\overline{\Omega}_A \mathbf{H})^I \times A)^+$ , where  $\lambda_{\mathbf{H}, \alpha_i}(\mathbf{q}_i) = \lambda_{\mathbf{H}}(\mathbf{q}_i)$  if  $\alpha_i = 0$ , and  $\lambda_{\mathbf{H}, \alpha_i}(\mathbf{q}_i) = I$  otherwise. Given a state  $\mathbf{v}$  of  $\mathcal{A}$ , the language associated to  $\mathbf{v}$  is the set  $\mathcal{L}(\mathbf{v})$  of all values of successful paths in  $\mathcal{A}_{\mathbf{v}}$ . The language associated to  $\mathcal{A}$ , denoted  $\mathcal{L}(\mathcal{A})$ , is the language associated to its root. Finally, the language associated to the pseudoword  $w$  is  $\mathcal{L}(w) = \mathcal{L}(\mathcal{T}(w))$ .

**Lemma 3.10.** *Let  $\mathcal{A}_1, \mathcal{A}_2$  be DRH-automata. Then, the languages  $\mathcal{L}(\mathcal{A}_1)$  and  $\mathcal{L}(\mathcal{A}_2)$  coincide if and only if the DRH-trees  $\vec{\mathcal{A}}_1$  and  $\vec{\mathcal{A}}_2$  are the same.*

*Proof.* Recall that, by Lemma 3.4, if  $\vec{\mathcal{A}}_1 = \vec{\mathcal{A}}_2$ , then  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are equivalent DRH-automata. Hence, Definition 3.3 makes clear the reverse implication. Conversely, let  $\mathcal{A}_i = \langle V_i, \rightarrow_i, \mathbf{q}_{i,0}, F_i, \lambda_{i,H}, \lambda_i \rangle$  ( $i = 1, 2$ ) be two DRH-automata such that  $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$ . We first observe that, for  $i = 1, 2$  and  $\alpha \in \Sigma^*$ , the state  $\mathbf{q}_{i,0}.\alpha$  is defined if and only if there exists an element in  $\mathcal{L}(\mathcal{A}_i)$  of the form  $(\alpha, \_, \_)$ . Hence, the state  $\mathbf{q}_{1,0}.\alpha$  is defined if and only if so is the state  $\mathbf{q}_{2,0}.\alpha$ . Choose  $\alpha = \alpha_0\alpha_1 \cdots \alpha_n \in \Sigma^*$ , with each  $\alpha_i \in \Sigma$ . If  $\mathbf{q}_{1,0}.\alpha \in F_1$ , then we have a successful path  $\mathbf{q}_{1,0} \xrightarrow{\alpha_0} \mathbf{q}_{1,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathbf{q}_{1,n+1}$ , so that, the element  $\prod_{i=0}^n (\alpha_i, \lambda_{1,H,\alpha_i}(\mathbf{q}_{1,i}), \lambda_1(\mathbf{q}_{1,i}))$  belongs to  $\mathcal{L}(\mathcal{A}_1)$  and hence, to  $\mathcal{L}(\mathcal{A}_2)$ . But that implies that, in  $\mathcal{A}_2$ , there is a successful path  $\mathbf{q}_{2,0} \xrightarrow{\alpha_0} \mathbf{q}_{2,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathbf{q}_{2,n+1}$ , which in turn yields that both  $\mathbf{q}_{1,0}.\alpha$  and  $\mathbf{q}_{2,0}.\alpha$  are terminal states. In particular the equalities in (1) hold. On the other hand, if  $\mathbf{q}_{1,0}.\alpha$  is not a terminal state, then condition (A.3) implies that  $\mathbf{q}_{1,0}.\alpha 0$  is defined. Since any DRH-automaton is trim, there exists  $\beta = \alpha_{n+2} \cdots \alpha_m \in \Sigma^*$  such that

$$(9) \quad \mathbf{q}_{1,0} \xrightarrow{\alpha_0} \mathbf{q}_{1,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathbf{q}_{1,n+1} \xrightarrow{0} \mathbf{q}_{1,n+2} \xrightarrow{\alpha_{n+2}} \cdots \xrightarrow{\alpha_m} \mathbf{q}_{1,m+1}$$

is a successful path in  $\mathcal{A}_1$ . Again, since  $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$ , this determines a successful path in  $\mathcal{A}_2$  given by  $\mathbf{q}_{2,0} \xrightarrow{\alpha_0} \mathbf{q}_{2,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathbf{q}_{2,n+1} \xrightarrow{0} \mathbf{q}_{2,n+2} \xrightarrow{\alpha_{n+2}} \cdots \xrightarrow{\alpha_m} \mathbf{q}_{2,m+1}$ , with the same value as the path (9). In particular, the  $(n+2)$ -nd letter (of the alphabet  $\Sigma \times (\overline{\Omega}_A H)^I \times A$ ) of that value is

$$(0, \lambda_{1,H,0}(\mathbf{q}_{1,n+1}), \lambda_1(\mathbf{q}_{1,n+1})) = (0, \lambda_{2,H,0}(\mathbf{q}_{2,n+1}), \lambda_2(\mathbf{q}_{2,n+1})).$$

But that means precisely that the desired equalities in (1) hold. Therefore,  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are equivalent and so,  $\vec{\mathcal{A}}_1 = \vec{\mathcal{A}}_2$ .  $\square$

**Proposition 3.11.** *Let  $u, v \in \overline{\Omega}_A S$ . Then the equality  $\rho_{\text{DRH}}(u) = \rho_{\text{DRH}}(v)$  holds if and only if  $\mathcal{L}(u) = \mathcal{L}(v)$  and  $H$  satisfies  $u = v$ .*

*Proof.* Let  $u$  and  $v$  be two equal pseudowords modulo DRH. In particular, the  $\mathcal{R}$ -classes  $[\rho_{\text{DRH}}(u)]_{\mathcal{R}}$  and  $[\rho_{\text{DRH}}(v)]_{\mathcal{R}}$  coincide and so, the DRH-trees  $\mathcal{T}(u)$  and  $\mathcal{T}(v)$  are the same, by Theorem 3.8. Therefore, we have  $\mathcal{L}(u) = \mathcal{L}(\mathcal{T}(u)) = \mathcal{L}(\mathcal{T}(v)) = \mathcal{L}(v)$ . As  $H$  is a subpseudovariety of DRH, we also have  $u =_H v$ . Conversely, suppose that  $\mathcal{L}(u) = \mathcal{L}(v)$  and  $u =_H v$ . By Lemma 3.10, it follows that  $\mathcal{T}(u) = \mathcal{T}(v)$ . Thus, by Theorem 3.8, the pseudovariety DRH satisfies  $u \mathcal{R} v$ . As, in addition, the pseudowords  $u$  and  $v$  are equal modulo  $H$ , we conclude by Lemma 2.8 that DRH satisfies  $u = v$ .  $\square$

#### 4. A CANONICAL FORM FOR $\kappa$ -WORDS OVER DRH

Throughout this section, we reserve the letter  $H$  to denote a pseudovariety of groups such that there exists a canonical form for the elements of  $\Omega_A^\kappa H$ . We denote by  $\text{cf}_H(w)$  the canonical form of  $w \in \Omega_A^\kappa H$  and set  $\text{cf}_H(I) = I$ . Our aim is to prove that this assumption on  $H$  is enough to define a canonical form for the elements of  $\Omega_A^\kappa \text{DRH}$ .

Given a finite DRH-automaton  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$  such that  $\lambda_H(V) \subseteq (\Omega_A^\kappa H)^I$ , let us define the expression  $\pi_{\text{cf}}(\mathcal{A})$  inductively on the number  $|V|$  of states as follows.

- If  $|V| = 1$ , then  $\mathcal{A} = \mathbf{1}$  and we take  $\pi_{\text{cf}}(\mathcal{A}) = I$ .
- If  $|V| > 1$  and  $\|\mathcal{A}\| < \infty$ , then we put

$$\pi_{\text{cf}}(\mathcal{A}) = \prod_{i=0}^{\|\mathcal{A}\|-1} \pi_{\text{cf}}(\mathcal{A}_{\mathbf{q}.1^i 0}) \text{cf}_H(\lambda_H(\mathbf{q}.1^i)) \lambda(\mathbf{q}.1^i).$$

- Finally, we suppose that  $|V| > 1$  and  $\|\mathcal{A}\| = \infty$ . Since  $\mathcal{A}$  is a finite automaton, we necessarily have a cycle of the form  $\mathbf{q}.1^\ell \xrightarrow{1} \mathbf{q}.1^{\ell+1} \xrightarrow{1} \dots \xrightarrow{1} \mathbf{q}.1^{\ell+n} \xrightarrow{1} \mathbf{q}.1^\ell$ , where  $\ell$  is a certain integer greater than or equal to  $\text{r.ind}(\mathcal{A})$ . Choose  $\ell$  to be the least possible. Then, we make  $\pi_{\text{cf}}(\mathcal{A})$  be given by

$$\begin{aligned} & \prod_{i=0}^{\text{r.ind}(\mathcal{A})-1} \pi_{\text{cf}}(\mathcal{A}_{\mathbf{q}.1^i 0}) \text{cf}_H(\lambda_H(\mathbf{q}.1^i)) \lambda(\mathbf{q}.1^i) \\ & \cdot \left( \prod_{i=\text{r.ind}(\mathcal{A})}^{\ell-1} \pi_{\text{cf}}(\mathcal{A}_{\mathbf{q}.1^i 0}) \text{cf}_H(\lambda_H(\mathbf{q}.1^i)) \lambda(\mathbf{q}.1^i) \right. \\ & \quad \left. \cdot \left( \prod_{i=0}^n \pi_{\text{cf}}(\mathcal{A}_{\mathbf{q}.1^{\ell+i} 0}) \text{cf}_H(\lambda_H(\mathbf{q}.1^{\ell+i})) \lambda(\mathbf{q}.1^{\ell+i}) \right)^\omega \right)^\omega. \end{aligned}$$

We point out that, by definition, the value of the  $\kappa$ -word over DRH naturally induced by  $\pi_{\text{cf}}(\mathcal{A})$  is precisely  $\pi(\mathcal{A})$ . On the other hand, it is easy to check that, for every  $w \in \overline{\Omega}_A \text{DRH}$ , if  $w \mathcal{R} \pi(\mathcal{A})$ , then the identity  $w = \pi(\mathcal{A}) \text{reg}(w)$  holds. Thus, in view of Theorem 3.8, we wish to standardize a choice of a finite DRH-automaton, say  $\mathcal{A}(w)$ , equivalent to  $\mathcal{T}(w)$ , for each  $w \in \Omega_A^\kappa \text{DRH}$ . After that, we may let the canonical form of  $w$  be given by  $\pi_{\text{cf}}(\mathcal{A}(w)) \text{cf}_H(\text{reg}(w))$ .

Fix a DRH-automaton  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$ . We say that two states  $\mathbf{v}_1, \mathbf{v}_2 \in V$  are *equivalent* if  $\pi(\mathcal{A}_{\mathbf{v}_1})$  and  $\pi(\mathcal{A}_{\mathbf{v}_2})$  lie in the same  $\mathcal{R}$ -class. Clearly, this defines an equivalence relation on  $V$ , say  $\sim$  (it should be clear from the context when we are referring to this equivalence relation or to the equivalence relation on  $\mathbb{A}_A$  introduced in Definition 3.3). We write  $[\mathbf{v}]$  for the equivalence class of  $\mathbf{v} \in V$ .

**Lemma 4.1.** *Let  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$  be a DRH-automaton and consider the equivalent class on  $V$  defined above. Then, for every  $\mathbf{v}_1, \mathbf{v}_2 \in V \setminus F$ , we have*

$$[\mathbf{v}_1] = [\mathbf{v}_2] \implies \begin{cases} [\mathbf{v}_1.0] = [\mathbf{v}_2.0] \text{ and } [\mathbf{v}_1.1] = [\mathbf{v}_2.1]; \\ \lambda_H(\mathbf{v}_1) = \lambda_H(\mathbf{v}_2) \text{ and } \lambda(\mathbf{v}_1) = \lambda(\mathbf{v}_2). \end{cases}$$

*Proof.* Let  $\mathbf{v}_1, \mathbf{v}_2 \in V \setminus F$  be non-terminal states. By definition, the classes  $[\mathbf{v}_1]$  and  $[\mathbf{v}_2]$  coincide if and only if  $\pi(\mathcal{A}_{\mathbf{v}_1}) \mathcal{R} \pi(\mathcal{A}_{\mathbf{v}_2})$ . Moreover, by Lemma

3.7, we have the equality  $\text{lbf}(\pi(\mathcal{A}_{v_1})) = (\pi(\mathcal{A}_{v_1.0})\lambda_H(v_1), \lambda(v_1), w_{1,r})$ , where  $w_{1,r}$  is  $\mathcal{R}$ -equivalent to  $\pi(\mathcal{A}_{v_1.1})$ . Similarly, there exists  $w_{2,r} \mathcal{R} \pi(\mathcal{A}_{v_2.1})$  such that  $\text{lbf}(\pi(\mathcal{A}_{v_2})) = (\pi(\mathcal{A}_{v_2.0})\lambda_H(v_2), \lambda(v_2), w_{2,r})$ . In particular, since we are assuming that  $\pi(\mathcal{A}_{v_1}) \mathcal{R} \pi(\mathcal{A}_{v_2})$ , the relations  $\pi(\mathcal{A}_{v_1.0}) \mathcal{R} \pi(\mathcal{A}_{v_2.0})$ , and  $\pi(\mathcal{A}_{v_1.1}) \mathcal{R} \pi(\mathcal{A}_{v_2.1})$  hold. But, that means that  $[v_1.0] = [v_2.0]$  and  $[v_1.1] = [v_2.1]$ . Also, the mid components of  $\text{lbf}(\pi(\mathcal{A}_{v_1}))$  and  $\text{lbf}(\pi(\mathcal{A}_{v_2}))$  should coincide, that is,  $\lambda(v_1) = \lambda(v_2)$ . Finally, we may derive the equality  $\lambda_H(v_1) = \lambda_H(v_2)$  as follows:

$$\begin{aligned} & \pi(\mathcal{A}_{v_1.0})\lambda_H(v_1) = \pi(\mathcal{A}_{v_2.0})\lambda_H(v_2) \quad \text{because } \pi(\mathcal{A}_{v_1}) \mathcal{R} \pi(\mathcal{A}_{v_2}) \\ \iff & \pi_{\text{irr}}(\mathcal{A}_{v_1.0})\text{id}(\mathcal{A}_{v_1.0})\lambda_H(v_1) = \pi_{\text{irr}}(\mathcal{A}_{v_2.0})\text{id}(\mathcal{A}_{v_2.0})\lambda_H(v_2) \quad \text{by (3)} \\ \implies & \text{id}(\mathcal{A}_{v_1.0})\lambda_H(v_1) = \text{id}(\mathcal{A}_{v_2.0})\lambda_H(v_2) \quad \text{by Lemma 3.7 and Proposition 2.3} \\ \implies & \lambda_H(v_1) = \lambda_H(v_2). \quad \square \end{aligned}$$

We define the *wrapping of a DRH-automaton*  $\mathcal{A} = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$  to be the DRH-automaton  $[\mathcal{A}] = \langle V/\sim, \rightarrow, [\mathbf{q}], F/\sim, \overline{\lambda}_H, \overline{\lambda} \rangle$ , where

- $[v].0 = [v.0]$  and  $[v].1 = [v.1]$ , for  $v \in V \setminus F$ ;
- $\overline{\lambda}_H([v]) = \lambda_H(v)$  and  $\overline{\lambda}([v]) = \lambda(v)$ , for  $v \in V$ .

By Lemma 6.3, this automaton is well defined. Furthermore, its definition ensures that  $\mathcal{A} \sim [\mathcal{A}]$ . The *wrapped DRH-automaton of*  $w \in \overline{\Omega}_A \text{DRH}$  is  $\mathcal{A}(w) = [\mathcal{T}(w)]$ . Observe that, by Lemmas 2.1 and 3.9, the label  $\lambda_H$  of  $\mathcal{T}(w)$  takes values in  $\Omega_A^\kappa \mathbf{H}$  when  $w$  is a  $\kappa$ -word. Our next goal is to prove that  $\mathcal{A}(w)$  is finite, provided  $w$  is a  $\kappa$ -word.

Let us associate to a pseudoword  $w \in (\overline{\Omega}_A \text{DRH})^I$  a certain set of its factors. For  $\alpha \in \Sigma^*$ , we define  $f_\alpha(w)$  inductively on  $|\alpha|$ :

$$\begin{aligned} f_\varepsilon(w) &= w; \\ (f_{\alpha 0}(w), a, f_{\alpha 1}(w)) &= \text{lbf}(f_\alpha(w)), \text{ for a certain } a \in A, \text{ whenever } f_\alpha(w) \neq I. \end{aligned}$$

Then, the set of DRH-factors of  $w$  is given by

$$\mathcal{F}(w) = \{f_\alpha(w) : \alpha \in \Sigma^* \text{ and } f_\alpha(w) \text{ is defined}\}.$$

The relevance of the definition of the set  $\mathcal{F}(w)$  is explained by the following result.

**Lemma 4.2.** *Let  $w \in \overline{\Omega}_A \text{DRH}$  and  $\mathcal{T}(w) = \langle V, \rightarrow, \mathbf{q}, F, \lambda_H, \lambda \rangle$ . Then, for every  $\alpha \in \Sigma^*$  such that  $f_\alpha(w)$  is defined, the relation  $f_\alpha(w) \mathcal{R} \pi(\mathcal{T}(w)_{\mathbf{q}, \alpha})$  holds.*

*Proof.* We prove the statement by induction on  $|\alpha|$ . When  $\alpha = \varepsilon$ , the result follows from Theorem 3.8. Let  $\alpha \in \Sigma^*$  and invoke the induction hypothesis to assume that  $f_\alpha(w)$  and  $\pi(\mathcal{T}(w)_{\mathbf{q}, \alpha})$  are  $\mathcal{R}$ -equivalents. Writing  $\text{lbf}(\pi(\mathcal{T}(w)_{\mathbf{q}, \alpha})) = (w_\ell, a, w_r)$ , Lemma 3.7 yields the relations  $w_\ell \mathcal{R} \pi(\mathcal{T}(w)_{\mathbf{q}, \alpha 0})$  and  $w_r \mathcal{R} \pi(\mathcal{T}(w)_{\mathbf{q}, \alpha 1})$ . On the other hand, since  $\text{lbf}(f_\alpha(w)) = (f_{\alpha 0}(w), b, f_{\alpha 1}(w))$ , using Lemma 2.7 we deduce that  $f_{\alpha 0} = w_\ell$ ,  $a = b$ , and  $f_{\alpha 1} \mathcal{R} w_r$ , leading to the desired conclusion.  $\square$

Hence, in order to prove that  $\mathcal{A}(w)$  is finite for every  $\kappa$ -word  $w$ , it suffices to prove that so is  $\mathcal{F}(w)/\mathcal{R}$ . The next two lemmas are useful to achieve that target.

**Lemma 4.3.** *Let  $w$  be a regular  $\kappa$ -word over DRH. Then, there exist  $\kappa$ -words  $x$ ,  $y$  and  $z$  over DRH such that  $w = xy^{\omega-1}z$ ,  $c(y) = c(w)$ ,  $\vec{c}(x) \subsetneq c(w)$ , and  $y$  is not regular.*

*Proof.* By definition of  $\kappa$ -word, we may write  $w = w_1 \cdots w_n$ , where each  $w_i$  is either a letter in  $A$  or an  $(\omega-1)$ -power of another  $\kappa$ -word. Since any letter of the cumulative content of  $w$  occurs in  $\text{lbf}_\infty(w)$  infinitely many times, there must be an  $(\omega-1)$ -power under which they all appear. Hence, since  $w$  is regular (and so,  $c(w) = \vec{c}(w)$ ), there exists an index  $i \in \{1, \dots, n\}$  such that  $w_i = v^{\omega-1}$  and  $c(v) = c(w)$ . Let  $j$  be the minimum such  $i$ . We have  $w = u_0 v_0^{\omega-1} z_0$ , where  $u_0 = w_1 \cdots w_{j-1}$ ,  $v_0^{\omega-1} = w_j$ , and  $z_0 = w_{j+1} \cdots w_n$ . Also, minimality of  $j$  yields that  $\vec{c}(u_0) \subsetneq \vec{c}(w) = c(w)$ . So, if  $v_0$  is not regular, then we just take  $x = u_0$ ,  $y = v_0$ , and  $z = z_0$ . Suppose that  $v_0$  is regular. Using the same reasoning, we may write  $v_0 = u_1 v_1^{\omega-1} z_1$ , with  $\vec{c}(u_1) \subsetneq c(w)$  and  $c(v_1) = c(v_0) = c(w)$ . Again, if  $v_1$  is not regular, then we may choose  $x = u_0 u_1$ ,  $y = v_1$  and  $z = z_1 v_0^{\omega-2} z_0$ . Otherwise, we repeat the process with  $v_1$ . Since  $w$  is a  $\kappa$ -word, there is only a finite number of occurrences of  $(\omega-1)$ -powers, so that, this iteration cannot run forever. Therefore, we eventually get  $\kappa$ -words  $x$ ,  $y$  and  $z$  satisfying the desired properties.  $\square$

**Lemma 4.4.** *Let  $w \in \Omega_A^\kappa \text{DRH}$  be regular. For each  $m \geq 1$ , let  $w'_m$  be the unique  $\kappa$ -word over DRH satisfying the equality  $w = \text{lbf}_1(w) \cdots \text{lbf}_m(w) w'_m$ . Then, both sets  $\{\text{lbf}_m(w) : m \geq 1\}$  and  $\{[w'_m]_{\mathcal{R}} : m \geq 1\}$  are finite.*

*Proof.* Write  $\text{lbf}_m(w) = w_m a_m$ , for every  $m \geq 1$ , and  $w = xy^{\omega-1}z$ , with  $x$ ,  $y$  and  $z$  satisfying the properties stated in Lemma 4.3. We define a sequence of pairs of possibly empty  $\kappa$ -words  $\{(u_i, v_i)\}_{i \geq 0}$  and a strictly increasing sequence of non-negative integers  $\{k_i\}_{i \geq 0}$  inductively as follows. We start with  $(u_0, v_0) = (I, x)$  and we let  $k_0$  be the maximum index such that  $\text{lbf}_1(w) \cdots \text{lbf}_{k_0}(w)$  is a prefix of  $x$ . If  $x$  has no prefix of this form, then we set  $k_0 = 0$ . We also write  $v_0 = v'_0 v''_0$ , with  $v'_0 = \text{lbf}_1(w) \cdots \text{lbf}_{k_0}(w)$  (by Proposition 2.3, given  $v'_0$  there is only one possible value for  $v''_0$ ). For each  $i \geq 0$ , we let  $u_{i+1}$  be such that  $w_{k_i+1} = v''_i u_{i+1}$  and  $v_{i+1}$  is such that  $y = u_{i+1} a_{k_i+1} v_{i+1}$ . Observe that, by uniqueness of first-occurrences factorizations, there is only one pair  $(u_{i+1}, v_{i+1})$  satisfying these conditions. The integer  $k_{i+1}$  is the maximum such that  $\text{lbf}_{k_i+2}(w) \cdots \text{lbf}_{k_{i+1}}(w)$  is a prefix of  $v_{i+1}$  (or  $k_{i+1} = k_i + 1$  if there is no such prefix) and we factorize  $v_{i+1} = v'_{i+1} v''_{i+1}$ , with  $v'_{i+1} = \text{lbf}_{k_i+2}(w) \cdots \text{lbf}_{k_{i+1}}(w)$ . By construction, for all  $i \geq 0$ , the pseudoidentity  $w'_{k_i+1} = v_{i+1} y^{\omega-(i+2)} z$  holds. In particular, for every  $m \geq 1$ , there exist  $i \geq 0$  and  $\ell \in \{2, \dots, k_{i+1} - k_i\}$  such that

$$(10) \quad w'_m = \text{lbf}_{k_i+\ell}(w) \text{lbf}_{k_i+\ell+1}(w) \cdots \text{lbf}_{k_{i+1}}(w) v''_{i+1} y^{\omega-(i+2)} z.$$

On the other hand, for all  $i \geq 0$ , the factorization  $y = u_{i+1}a_{k_i+1}v_{i+1}$  is such that  $a_{k_i+1} \notin c(u_{i+1})$  (recall that  $a_{k_i+1} \notin c(w_{k_i+1})$  and  $u_{i+1}$  is a factor of  $w_{k_i+1}$ ). By uniqueness of first-occurrences factorization over DRH, it follows that the set  $\{(u_i, v_i)\}_{i \geq 0}$  is finite. Consequently, the set  $\{\text{lbf}_{k_i+\ell}(w)\text{lbf}_{k_i+\ell+1}(w) \cdots \text{lbf}_{k_{i+1}}(w)v''_{i+1} : i \geq 0, \ell \in \{2, \dots, k_{i+1} - k_i\}\}$  is also finite. In particular, there is only a finite number of  $\kappa$ -words  $\text{lbf}_m(w)$ . Finally, taking into account that  $c(z) \subseteq c(y)$  and (10) we may conclude that there are only finitely many  $\mathcal{R}$ -classes of the form  $[w'_m]_{\mathcal{R}}$  ( $m \geq 1$ ).  $\square$

Now, we are able to prove that  $\mathcal{F}(w)/\mathcal{R}$  is finite for every  $\kappa$ -word  $w$  over DRH.

**Proposition 4.5.** *Let  $w$  be a possibly empty  $\kappa$ -word over DRH. Then, the quotient  $\mathcal{F}(w)/\mathcal{R}$  is finite.*

*Proof.* We prove the result by induction on  $|c(w)|$ . If  $|c(w)| = 0$ , then it is trivial. Suppose that  $|c(w)| \geq 1$ . We distinguish two possible scenarios.

**Case 1.:** The  $\kappa$ -word  $w$  is not regular, that is,  $\tilde{c}(w) \subsetneq c(w)$ .

Then, there exists  $k \geq 1$  such that  $w = w_1a_1 \cdots w_ma_mw'_m$ , with  $\text{lbf}_k(w) = w_ka_k$ , for  $k = 1, \dots, m$  and  $c(w'_m) \subsetneq c(w)$ . By definition of  $f_\alpha(w)$ , we have the identities  $f_{1^{k-1}0}(w) = w_k$  (for  $k = 1, \dots, m$ ) and  $f_{1^m} = w'_m$ . Hence, we may deduce that  $\mathcal{F}(w)$  is the union of the sets  $\mathcal{F}(w_k)$  (for  $k = 1, \dots, m$ ) together with  $\mathcal{F}(w'_m)$ . Using the induction hypothesis on each one of the intervening sets, we conclude that  $\mathcal{F}(w)$  is finite.

**Case 2.:** The  $\kappa$ -word  $w$  is regular.

Again, write  $\text{lbf}_k(w) = w_ka_k$  and  $w = \text{lbf}_1(w) \cdots \text{lbf}_k(w)w'_k$ , for  $k \geq 1$ . Since  $f_{1^{k-1}0}(w) = w_k$  and  $f_{1^k}(w) = w'_k$ , for every  $k \geq 1$ , by Lemma 4.4, we know that the sets  $\{f_{1^{k-1}0}(w)\}_{k \geq 1}$  and  $\{[f_{1^k}(w)]_{\mathcal{R}}\}_{k \geq 1}$  are both finite. Applying the induction hypothesis to each factor  $w_k$ , we derive that  $\{[f_{1^{k-1}0\alpha}(w)]_{\mathcal{R}} : \alpha \in \Sigma^*\}_{k \geq 1}$  is also a finite set. Therefore, since any element of  $\mathcal{F}(w)/\mathcal{R}$  is of one of the forms  $[f_{1^{k-1}0\alpha}(w)]_{\mathcal{R}}$  and  $[f_{1^k}(w)]_{\mathcal{R}}$ , we conclude that  $\mathcal{F}(w)/\mathcal{R}$  is finite as well.  $\square$

As an immediate consequence (recall Lemma 4.2), we obtain:

**Corollary 4.6.** *Let  $w$  be a possibly empty  $\kappa$ -word. Then, the wrapped DRH-automaton  $\mathcal{A}(w)$  is finite.*  $\square$

Unlike the aperiodic case R, the converse of Corollary 4.6 does not hold in general. For instance, taking  $H = G$ , it is not hard to see that  $\mathcal{A}(a^{p^\omega}b)$  (with  $p$  a prime number) is finite, although  $a^{p^\omega}b$  is not a  $\kappa$ -word over DRG. A converse is achieved when we further require that the labels  $\lambda_H$  are valued by  $\kappa$ -words over  $H$  and that  $\rho_H(\text{reg}(w))$  is itself a  $\kappa$ -word.

For a given  $w \in (\Omega_A^\kappa \text{DRH})^I$ , the expression

$$\text{cf}(w) = \pi_{\text{cf}}(\mathcal{A}(w))\text{cf}_H(\rho_H(\text{reg}(w)))$$



is said the *canonical form* of  $w$ . We write  $\text{cf}(u) \equiv \text{cf}(v)$  (with  $u, v \in (\Omega_A^\kappa \text{DRH})^I$ ) when both sides coincide. We have just proved the claimed existence of a canonical form for elements of  $\Omega_A^\kappa \text{DRH}$ .

**Theorem 4.7.** *Let  $\mathbf{H}$  be a pseudovariety of groups such that there exists a canonical form for the elements of  $\Omega_A^\kappa \mathbf{H}$ , say  $\text{cf}_{\mathbf{H}}(-)$ . Then, for all  $\kappa$ -words  $u$  and  $v$  over  $\text{DRH}$ , the equality  $u = v$  holds if and only if  $\text{cf}(u) \equiv \text{cf}(v)$ .  $\square$*

## 5. $\bar{\kappa}$ -TERMS SEEN AS WELL-PARENTHEORIZED WORDS

In Section 3, we characterized  $\mathcal{R}$ -classes over  $\text{DRH}$  by means of certain equivalence classes of automata. In order to solve the  $\kappa$ -word problem over  $\text{DRH}$ , the next goal is to find an algorithm to construct such automata. This section serves the purpose of preparing that construction.

**5.1. General definitions.** Let  $B$  be a possibly infinite alphabet and consider the associated alphabet  $B_{[]} = B \uplus \{[{}^q, ]^q : q \in \mathbb{Z}\}$ . We say that a word in  $B_{[]}^*$  is *well-parenthesized over  $B$*  if it does not contain  $[{}^q, ]^q$  as a factor and if it can be reduced to the empty word  $\varepsilon$  by applying the rewriting rules  $[{}^q, ]^q \rightarrow \varepsilon$  and  $a \rightarrow \varepsilon$ , for  $q \in \mathbb{Z}$  and  $a \in B$ . We denote the set of all well-parenthesized words over  $B$  by  $\text{Dyck}(B)$ . The *content* of a well-parenthesized word  $x$  is the set of letters in  $B$  that occur in  $x$  and it is denoted  $c(x)$ .

To each  $\bar{\kappa}$ -term we may associate a well-parenthesized word over  $A$  inductively as follows:

$$\begin{aligned} \text{word}(a) &= a, & \text{if } a \in A; \\ \text{word}(u \cdot v) &= \text{word}(u)\text{word}(v), & \text{if } u \text{ and } v \text{ are } \bar{\kappa}\text{-terms}; \\ \text{word}(u^{\omega+q}) &= [{}^q \text{word}(u)]^q, & \text{if } u \text{ is a } \bar{\kappa}\text{-term}. \end{aligned}$$

Conversely, we associate a  $\kappa$ -word to each well-parenthesized word over  $A$  as follows:

$$\begin{aligned} \text{om}(a) &= a, & \text{if } a \in A; \\ \text{om}(xy) &= \text{om}(x) \cdot \text{om}(y), & \text{if } x, y \in \text{Dyck}(A); \\ \text{om}([{}^q x]^q) &= \text{om}(x)^{\omega+q}, & \text{if } x \in \text{Dyck}(A). \end{aligned}$$

Note that, due to the associative property in both  $\text{Dyck}(A)$  and  $\Omega_A^\kappa \mathbf{S}$ ,  $\text{om}(-)$  is well-defined. With the aim of distinguishing the occurrences of each letter in  $A$  in a well-parenthesized word  $x$  over  $A$ , we assign to each  $x \in \text{Dyck}(A)$  a well-parenthesized word  $x_{\mathbb{N}}$  over  $A \times \mathbb{N}$  containing all the information about the position of the letters. With that in mind we define recursively the following family of functions  $\{p_k : \text{Dyck}(A) \rightarrow \text{Dyck}(A \times \mathbb{N})\}_{k \geq 0}$ :

$$\begin{aligned} p_k(a) &= (a, k+1), & \text{if } a \in A; \\ p_k([{}^q, ]^q) &= [{}^q \text{ and } p_k([{}^q, ]^q) = ]^q, & \text{if } q \in \mathbb{Z}; \\ p_k(ay) &= p_k(a)p_{k+1}(y), & \text{if } a \in A_{[]} \text{ and } y \in A_{[]}^*. \end{aligned}$$

We define  $x_{\mathbb{N}} = p_0(x)$ . For instance, if  $x = a[a^qb[r^rc]^r]^q$ , then  $x_{\mathbb{N}}$  is the word  $(a, 1)[^q(b, 2)[^r(c, 3)]^r]^q$ . It is often convenient to denote the pair  $(a, i)$  by  $a_i$ . Let  $x \in \text{Dyck}(A \times \mathbb{N})$ . Then, we may associate to  $x$  two well-parenthesized words  $\pi_A(x)$  and  $\pi_{\mathbb{N}}(x)$  corresponding to the projection of  $x$  onto  $A_{[]}^*$  and onto  $\mathbb{N}_{[]}^*$ , respectively. We denote  $c_A(x) = c(\pi_A(x))$  and  $c_{\mathbb{N}}(x) = c(\pi_{\mathbb{N}}(x))$ . Given a  $\bar{\kappa}$ -term  $w$ , we denote by  $\bar{w}$  the well-parenthesized word  $0_0\text{word}(w\#)_{\mathbb{N}}$  over the alphabet  $(A \uplus \{0, \#\}) \times \mathbb{N}$ . The map  $\eta : \text{Dyck}(A \times \mathbb{N}) \rightarrow \Omega_A^{\kappa} \mathcal{S}$  assigns to each well-parenthesized word  $x \in \text{Dyck}(A \times \mathbb{N})$  the  $\kappa$ -word  $\eta(x) = \text{om}(\pi_A(x))$ .

Let  $x$  be a well-parenthesized word over  $A \times \mathbb{N}$ . We define its *tail*  $\mathbf{t}_i(x)$  from position  $i \in \mathbb{N}$  inductively as follows

$$\begin{aligned} \mathbf{t}_i(\varepsilon) &= \varepsilon; \\ \mathbf{t}_i(yz) &= \mathbf{t}_i(z), \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } i \notin c_{\mathbb{N}}(y); \\ \mathbf{t}_i(a_i y) &= y, \quad \text{if } y \in \text{Dyck}(A \times \mathbb{N}); \\ \mathbf{t}_i([{}^q y]^q z) &= \mathbf{t}_i(y)[{}^{q-1} y]^{q-1} z, \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } i \in c_{\mathbb{N}}(y). \end{aligned}$$

The *prefix* of  $x \in \text{Dyck}(A \times \mathbb{N})$  until  $a \in A$  is defined by

$$\begin{aligned} \mathbf{p}_a(\varepsilon) &= \varepsilon; \\ \mathbf{p}_a(yz) &= y\mathbf{p}_a(z), \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } a \notin c_A(y); \\ \mathbf{p}_a(a_i y) &= \varepsilon, \quad \text{if } y \in \text{Dyck}(A \times \mathbb{N}); \\ \mathbf{p}_a([{}^q y]^q z) &= \mathbf{p}_a(y), \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } a \in c_A(y). \end{aligned}$$

The *factor* of a well-parenthesized word  $x \in \text{Dyck}(A \times \mathbb{N})$  from  $i \in \mathbb{N}$  until  $a \in A$  is given by

$$x(i, a) = \mathbf{p}_a(\mathbf{t}_i(x)).$$

If instead, we are given a  $\bar{\kappa}$ -term  $w$ , then we write  $w(i, a)$  to mean the  $\kappa$ -word  $\eta(\bar{w}(i, a))$ . If  $a$  is a letter occurring in  $\pi_A(x)$ , for a well-parenthesized word  $x$  over  $A \times \mathbb{N}$ , then it is possible to write  $x = ya_i z$  with  $y$  and  $z$  possibly empty not necessarily well-parenthesized words over  $A \times \mathbb{N}$  such that  $a \notin c_A(y)$ . In this case we say that  $a_i$  is a *marker* of  $x$ . If  $a_i$  is the last first occurrence of a letter, that is, if the inclusion  $c_A(z) \subseteq c_A(ya_i)$  holds, then we say that  $a_i$  is the *principal marker* of  $x$ .

## 5.2. Properties of tails and prefixes of well-parenthesized words.

The next results state some properties concerning tails and prefixes of well-parenthesized words. Some of the proofs are omitted since they are rather technical and entirely similar to the proofs of the analogous results in [5]. When that is the case, we refer the reader to the corresponding result.

**Lemma 5.1** (cf. [5, Lemma 5.3]). *Let  $x \in \text{Dyck}(A \times \mathbb{N})$  and let  $a, b \in A$ . If  $b \in c_A(\mathbf{p}_a(x))$ , then  $\mathbf{p}_b(\mathbf{p}_a(x)) = \mathbf{p}_b(x)$ .*

**Lemma 5.2** (cf. [5, Lemma 5.4]). *Let  $x \in \text{Dyck}(A \times \mathbb{N})$  be such that  $a$  belongs to  $c_A(x)$ . If  $k \in c_{\mathbb{N}}(\mathbf{p}_a(x))$ , then  $a \in c_A(\mathbf{t}_k(x))$ .*

**Lemma 5.3** (cf. [5, Lemma 5.5]). *Let  $x \in \text{Dyck}(A \times \mathbb{N})$  and let  $k \in c_{\mathbb{N}}(\mathfrak{p}_a(x))$ . Then, we have  $\mathfrak{t}_k(\mathfrak{p}_a(x)) = \mathfrak{p}_a(\mathfrak{t}_k(x))$ .*

**Lemma 5.4.** *Let  $\vec{x} = (x_j)_{j \geq 0}$  and  $\vec{y} = (y_j)_{j \geq 0}$  be two sequences of possibly empty well-parenthesized words over  $A \times \mathbb{N}$  such that  $x_0 y_0 \neq \varepsilon$ , and for every  $i, j \geq 0$ , the index  $i$  occurs in  $\pi_{\mathbb{N}}(x_0 y_0 x_1 y_1 \cdots x_j y_j)$  at most once. Let  $\vec{q} = (q_j)_{j \geq 0}$  be a sequence of integers. For each  $n \geq 0$ , we define the well-parenthesized words  $\mu_n(\vec{x}, \vec{y}, \vec{q})$  and  $\xi_n(\vec{x}, \vec{y}, \vec{q})$  as follows:*

$$\begin{aligned} \mu_0(\vec{x}, \vec{y}, \vec{q}) &= x_0 y_0 \\ \mu_{n+1}(\vec{x}, \vec{y}, \vec{q}) &= x_{n+1} [\mu_n(\vec{x}, \vec{y}, \vec{q})]^{q_n} y_{n+1}, \text{ if } n \geq 0 \\ \xi_n(\vec{x}, \vec{y}, \vec{q}) &= [\mu_n(\vec{x}, \vec{y}, \vec{q})]^{q_n-1} y_{n+1}, \text{ if } n \geq 0. \end{aligned}$$

Let  $i$  be a natural number and suppose that  $i \in c_{\mathbb{N}}(x_\ell y_\ell)$  for a certain  $\ell \geq 0$ . Then, for every  $n \geq \ell$ , the following equality holds:

$$(11) \quad \mathfrak{t}_i(\mu_n(\vec{x}, \vec{y}, \vec{q})) = \mathfrak{t}_i(\mu_\ell(\vec{x}, \vec{y}, \vec{q})) \cdot \xi_\ell(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}).$$

*Proof.* We argue by induction on  $n$ . If  $n = \ell$ , then the result holds clearly, since the factor  $\xi_\ell(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})$  vanishes in (11). Suppose that  $n > \ell$  and that the result holds for any smaller  $n$ . We may compute

$$\begin{aligned} \mathfrak{t}_i(\mu_n(\vec{x}, \vec{y}, \vec{q})) &= \mathfrak{t}_i(x_n [\mu_{n-1}(\vec{x}, \vec{y}, \vec{q})]^{q_{n-1}} y_n) \\ &= \mathfrak{t}_i(\mu_{n-1}(\vec{x}, \vec{y}, \vec{q})) \cdot [\mu_{n-1}(\vec{x}, \vec{y}, \vec{q})]^{q_{n-1}-1} y_n \\ &\quad \text{since } i \notin c_{\mathbb{N}}(x_n) \text{ and } i \in c_{\mathbb{N}}(\mu_{n-1}(\vec{x}, \vec{y}, \vec{q})) \\ &= \mathfrak{t}_i(\mu_{n-1}(\vec{x}, \vec{y}, \vec{q})) \cdot \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}) \\ &= \mathfrak{t}_i(\mu_\ell(\vec{x}, \vec{y}, \vec{q})) \quad \text{by induction hypothesis} \\ &\quad \cdot \xi_\ell(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-2}(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}) \end{aligned}$$

obtaining the desired equality (11).  $\square$

By successively applying Lemma 5.4, we obtain the following:

**Corollary 5.5.** *Using the same notation and assuming the same hypothesis as in the previous lemma, suppose that  $k \in c_{\mathbb{N}}(y_0)$ . Then,*

(a) *if  $i \in c_{\mathbb{N}}(x_\ell)$  for a certain  $\ell \geq 0$ , then the equality*  

$$\mathfrak{t}_k(\mathfrak{t}_i(\mu_n(\vec{x}, \vec{y}, \vec{q}))) = \mathfrak{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-2}(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})$$
  
*holds for every  $n \geq \ell$ ;*

(b) *if  $i \in c_{\mathbb{N}}(y_\ell)$  for a certain  $\ell \geq 1$ , then the equality*

$$\begin{aligned} \mathfrak{t}_k(\mathfrak{t}_i(\mu_n(\vec{x}, \vec{y}, \vec{q}))) &= \mathfrak{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q}) \\ &\quad \cdot [\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q_\ell-2} y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}) \end{aligned}$$

*holds for every  $n \geq \ell$ .*  $\square$

The reader may wish to compare the next result with [5, Lemma 5.8].

**Lemma 5.6.** *Let  $w$  be a  $\bar{\kappa}$ -term,  $i \geq 0$ , and  $a \in c(w)$ . Assume that  $b_k$  is the principal marker of  $\bar{w}(i, a)$ . Then, the following properties hold:*

- (a)  $\mathbf{p}_b(\bar{w}(i, a)) = \bar{w}(i, b)$ ;
- (b) DRH satisfies  $\eta(\mathbf{t}_k(\bar{w}(i, a))) \mathcal{R} w(k, a)$ .

Moreover, if the projection of  $w(i, a)$  onto  $\bar{\Omega}_A \text{DRH}$  is not regular, then the relation in (b) becomes an equality in  $\bar{\Omega}_A S$ .

*Proof.* By definition, we have  $\bar{w}(i, a) = \mathbf{p}_a(\mathbf{t}_i(\bar{w}))$ . Since  $b \in c_A(\bar{w}(i, a))$ , it follows from Lemma 5.1 that  $\mathbf{p}_b(\bar{w}(i, a)) = \mathbf{p}_b(\mathbf{p}_a(\mathbf{t}_i(\bar{w}))) = \mathbf{p}_b(\mathbf{t}_i(\bar{w})) = \bar{w}(i, b)$ .

Let us prove the second assertion. By definition of  $\bar{w}$ , we know that  $b_k$  appears exactly once in  $\bar{w}$  and the same happens with the index  $i$ . Let  $\bar{w} = x \cdot b_k \cdot y$ . We distinguish the cases where  $x$  and  $y$  are both possibly empty well-parenthesized words and where neither of  $x$  nor  $y$  is a well-parenthesized word. In the first case, since  $b_k \in c(\bar{w}(i, a)) \subseteq c(\mathbf{t}_i(\bar{w}))$ , the index  $i$  must belong to  $c_{\mathbb{N}}(x)$ . So, we get  $\mathbf{t}_k(\bar{w}(i, a)) = \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(\bar{w}))) = \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(x)b_k y))$ . Should  $a$  occur in  $\mathbf{t}_i(x)b_k$ , then  $b_k$  would not appear in  $\bar{w}(i, a)$ . So, it follows that

$$(12) \quad \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(x)b_k y)) = \mathbf{t}_k(\mathbf{t}_i(x)b_k \mathbf{p}_a(y)) = \mathbf{p}_a(y).$$

On the other hand, we have the equalities  $\bar{w}(k, a) = \mathbf{p}_a(\mathbf{t}_k(\bar{w})) = \mathbf{p}_a(y) \stackrel{(12)}{=} \mathbf{t}_k(\bar{w}(i, a))$ , and so the desired relation (b) follows.

Now, we suppose that

$$\begin{aligned} x &= x_n [^{q_{n-1}} x_{n-1} \cdots [^{q_1} x_1 [^{q_0} x_0, \\ b_k y &= y_0 ]^{q_0} y_1 ]^{q_1} \cdots y_{n-1} ]^{q_{n-1}} y_n, \end{aligned}$$

where all the  $x_j$ 's and  $y_j$ 's are possibly empty well-parenthesized words, for  $j = 0, \dots, n$ . We note that, since  $k \in c_{\mathbb{N}}(\bar{w}(i, a)) = c_{\mathbb{N}}(\mathbf{p}_a(\mathbf{t}_i(\bar{w})))$ , Lemma 5.3 yields the equalities

$$(13) \quad \mathbf{t}_k(\bar{w}(i, a)) = \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(\bar{w}))) = \mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\bar{w}))).$$

With that in mind, we start by computing the elements  $\mathbf{t}_k(\bar{w})$  and  $\mathbf{t}_k(\mathbf{t}_i(\bar{w}))$ . Let

$$\begin{aligned} \vec{x} &= (x_0, x_1, \dots, x_n, \varepsilon, \varepsilon, \dots); \\ \vec{y} &= (y_0, y_1, \dots, y_n, \varepsilon, \varepsilon, \dots); \\ \vec{q} &= (q_0, q_1, \dots, q_{n-1}, 0, 0, \dots) \end{aligned}$$

and let  $\ell \in \{0, 1, \dots, n\}$  be such that  $i \in c_{\mathbb{N}}(x_\ell y_\ell)$ . Noticing that  $\bar{w} = \mu_n(\vec{x}, \vec{y}, \vec{q})$ ,  $k$  belongs to  $c_{\mathbb{N}}(y_0)$ , and using Lemma 5.4 we obtain

$$\begin{aligned} \mathbf{t}_k(\bar{w}) &= \mathbf{t}_k(\mu_0(\vec{x}, \vec{y}, \vec{q})) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}) \\ (14) \quad &= \mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}) \end{aligned}$$

Now, we have two possible situations.

- (i)  $i \in c_{\mathbb{N}}(x_\ell)$ , for a certain  $\ell \in \{0, \dots, n\}$ ;

(ii)  $i \in c_{\mathbb{N}}(y_\ell)$ , for a certain  $\ell \in \{n, \dots, 1\}$ .

If we are in Case (i), then we may use Corollary 5.5(a) and get

$$\mathbf{t}_k(\mathbf{t}_i(\overline{w})) = \mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-2}(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}).$$

Hence, we have an equality between  $\mathbf{t}_k(\overline{w}(i, a)) = \mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\overline{w})))$  and  $\overline{w}(k, a) = \mathbf{p}_a(\mathbf{t}_k(\overline{w}))$ , thereby proving (b).

On the other hand, when the situation occurring is (ii), Corollary 5.5(b) yields

$$\begin{aligned} \mathbf{t}_k(\mathbf{t}_i(\overline{w})) &= \mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q}) \\ &\quad \cdot [{}^{q\ell-2}\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q\ell-2} y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}). \end{aligned}$$

If the first occurrence of  $a$  in  $\mathbf{t}_k(\mathbf{t}_i(\overline{w}))$  is in

$$\mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q})$$

or in  $\mu_\ell(\vec{x}, \vec{y}, \vec{q})$ , then the first occurrence of  $a$  in  $\mathbf{t}_k(\overline{w})$  is also in one of these factors and we easily conclude that

$$\begin{aligned} \mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\overline{w}))) &= \mathbf{p}_a(\mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q}) \cdot \mu_\ell(\vec{x}, \vec{y}, \vec{q})) \\ &= \mathbf{p}_a(\mathbf{t}_k(\overline{w})), \end{aligned}$$

thereby proving again an equality in (b).

Otherwise, the first occurrence of  $a$  in  $\mathbf{t}_k(\mathbf{t}_i(\overline{w}))$  is in

$$y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}).$$

Analyzing the equality (14), we deduce that  $a$  occurs for the first time in  $\mathbf{t}_k(\overline{w})$  also in the factor  $y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})$ . Then, we may compute

$$\begin{aligned} \mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\overline{w}))) &= \mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q}) \\ (15) \quad &\quad \cdot [{}^{q\ell-2}\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q\ell-2} \cdot \mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})) \end{aligned}$$

$$\begin{aligned} \mathbf{p}_a(\mathbf{t}_k(\overline{w})) &= \mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q}) \\ (16) \quad &\quad \cdot [{}^{q\ell-1}\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q\ell-1} \cdot \mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})). \end{aligned}$$

Moreover, using again Lemma 5.4, we obtain

$$\begin{aligned} \overline{w}(i, a) &= \mathbf{p}_a(\mathbf{t}_i(\overline{w})) = \mathbf{p}_a(\mathbf{t}_i(\mu_n(\vec{x}, \vec{y}, \vec{q}))) \\ &= \mathbf{p}_a(\mathbf{t}_i(y_\ell)) \cdot \xi_\ell(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}) \\ &= \mathbf{t}_i(y_\ell) [{}^{q\ell-1}\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q\ell-1} \mathbf{p}_a(y_{\ell+1} \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})) \\ &= \mathbf{t}_i(y_\ell) [{}^{q\ell-1}x_\ell [{}^{q\ell-1}x_{\ell-1} [{}^{q\ell-2} \dots [{}^{q_0} x_0 y_0]^{q_0} \dots]^{q\ell-2} y_{\ell-1}]^{q\ell-1} y_\ell]^{q\ell-1} \\ (17) \quad &\quad \cdot \mathbf{p}_a(y_{\ell+1} \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})) \end{aligned}$$

Since  $b_k$  is the principal marker of  $\overline{w}(i, a)$ , we know that the following inclusion holds:

$$c_A(y_0 y_1 \cdots y_\ell \cdot \mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}))) \subseteq c_A(\mathbf{t}_i(y_\ell) x_\ell \cdots x_0 b_k).$$

Also, by definition of  $\mu_\ell(\vec{x}, \vec{y}, \vec{q})$ , we have an inclusion

$$c_A(\mathbf{t}_i(y_\ell)x_\ell \cdots x_0b_k) \subseteq c_A(\mu_\ell(\vec{x}, \vec{y}, \vec{q})).$$

Consequently, we obtain

$$c_A(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}))) \subseteq c_A(\mu_\ell(\vec{x}, \vec{y}, \vec{q})).$$

Observing that

$$(18) \quad \vec{c}(\eta([\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q_\ell-2})) = c(\eta(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q})))),$$

we end up with the desired relations, which are valid in DRH:

$$\begin{aligned} \eta(\mathbf{t}_k(\overline{w}(i, a))) &\stackrel{(13), (15)}{=} \eta(\mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q})) \\ &\quad \cdot \eta([\mu_\ell(\vec{x}, \vec{y}, \vec{q})]^{q_\ell-2}) \\ &\quad \cdot \eta(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}))) \\ &\stackrel{(18)}{\mathcal{R}} \eta(\mathbf{t}_k(y_0) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{\ell-1}(\vec{x}, \vec{y}, \vec{q})) \\ &\quad \cdot \eta(\mu_\ell(\vec{x}, \vec{y}, \vec{q}))^{\omega+q_\ell-1} \\ &\quad \cdot \eta(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1}(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}))) \\ &\stackrel{(16)}{=} \eta(\overline{w}(k, a)) = w(k, a). \end{aligned}$$

We finally observe that we actually proved an equality in  $\overline{\Omega}_A \mathbf{S}$  rather than a relation modulo DRH, except in the last situation. But that scenario only occurs when  $w(i, a)$  is regular modulo DRH. Indeed, since  $b_k \in c(y_0)$  is the principal marker of  $w(i, a)$ , from the equality (17), we may deduce that  $\vec{c}(w(i, a)) = c(w(i, a))$ , which by Proposition 2.4 implies that  $\rho_{\text{DRH}}(w(i, a))$  is regular.  $\square$

For a well-parenthesized word  $x$  over  $A \times \mathbb{N}$ , we consider the following property:

$$(H(x)) \quad \forall a, b \in A, \quad \forall i \in \mathbb{N}, \quad a_i, b_i \in c(x) \implies a = b$$

The proof of the next result may be easily adapted from the proof of [5, Lemma 5.9].

**Lemma 5.7.** *Let  $x \in \text{Dyck}(A \times \mathbb{N}) \setminus \{\varepsilon\}$  satisfy  $(H(x))$  and suppose that  $a_i$  is a marker of  $x$ . Then the equality  $\eta(x) = \eta(\mathbf{p}_a(x) \cdot a_i \cdot \mathbf{t}_i(x))$  holds.*

**Corollary 5.8.** *Let  $w$  be a  $\overline{\kappa}$ -term. Let  $i \in \mathbb{N}$  and  $a \in A \uplus \{\#\}$ , and let  $b_k$  be the principal marker of  $\overline{w}(i, a)$ . Suppose that  $\text{lbf}(w(i, a)) = (w_\ell, m, w_r)$ . Then,  $m = b$  and DRH satisfies  $w_\ell = w(i, b)$ , and  $w_r \mathcal{R} w(k, a)$ . Moreover, if  $\rho_{\text{DRH}}(w(i, a))$  is not regular, then  $\text{lbf}(w(i, a)) = (w(i, b), b, w(k, a))$ .*

*Proof.* As  $b_k$  is the principal marker of  $\overline{w}(i, a)$ , we can write  $\overline{w}(i, a) = xb_ky$ , where  $c_A(y) \subseteq c_A(xb_k)$  and  $b \notin c_A(x)$ . Since  $(H(x))$  holds, Lemma 5.7 yields  $\eta(\overline{w}(i, a)) = \eta(\mathbf{p}_b(\overline{w}(i, a)) \cdot b_k \cdot \mathbf{t}_k(\overline{w}(i, a))) = \eta(\mathbf{p}_b(\overline{w}(i, a))) \cdot b \cdot \eta(\mathbf{t}_k(\overline{w}(i, a)))$ .

Furthermore, since  $b \notin c_A(x)$ , we also have  $c_A(\mathbf{p}_b(\overline{w}(i, a))) = c_A(x)$  and consequently, the left basic factorization of  $w(i, a)$  is precisely

$$(\eta(\mathbf{p}_b(\overline{w}(i, a))), b, \eta(\mathbf{t}_k(\overline{w}(i, a)))).$$

In particular, we have  $m = b$  and, by Lemma 5.6, the pseudovariety DRH satisfies  $w_\ell = w(i, b)$  and  $w_r \mathcal{R} w(k, a)$ , with an equality in § in the latter relation when  $w(i, a)$  is not regular modulo DRH.  $\square$

## 6. DRH-GRAPHS AND THEIR COMPUTATION

We begin this section with the definition of a DRH-graph. Through these structures, we are able to decide whether two  $\kappa$ -words are  $\mathcal{R}$ -equivalent over DRH. If we further assume that the word problem is decidable in  $\Omega_A^\kappa \mathbf{H}$ , then the word problem is decidable in  $\Omega_A^\kappa \text{DRH}$  as well.

**Definition 6.1.** *Let  $w$  be a  $\overline{\kappa}$ -term. The DRH-graph of  $w$  is the finite DRH-automaton*

$$\mathcal{G}(w) = \langle V(w), \rightarrow, \mathbf{q}(0, \#), \{\varepsilon\}, \lambda_{\mathbf{H}}, \lambda \rangle,$$

*defined as follows. The set of states is*

$$V(w) = \{\mathbf{q}(i, a) : 0 \leq i < |\overline{w}|, a \in c_A(\overline{w}) \text{ and } w(i, a) \neq I\} \cup \{\varepsilon\}.$$

*Let  $\mathbf{q}(i, a) \in V(w) \setminus \{\varepsilon\}$  and  $b_k$  be the principal marker of  $\overline{w}(i, a)$ . The transitions of  $\mathbf{q}(i, a)$  are  $\mathbf{q}(i, a).0 = \mathbf{q}(i, b)$  and  $\mathbf{q}(i, a).1 = \mathbf{q}(k, a)$ . The labels are  $\lambda_{\mathbf{H}}(\mathbf{q}(i, a)) = \rho_{\mathbf{H}}(\text{reg}(w(i, b)))$  and  $\lambda(\mathbf{q}(i, a)) = b$ . If a state  $\mathbf{q}(i, a)$  is not reached from the root  $\mathbf{q}(0, \#)$ , then we discard it from  $V(w)$ .*

The following result suggests that the construction of  $\mathcal{G}(w)$  might be a starting point to solve the  $\kappa$ -word problem over DRH algorithmically.

**Proposition 6.2.** *For every  $\overline{\kappa}$ -term  $w$ ,  $\mathcal{G}(w)$  is a DRH-automaton equivalent to  $\mathcal{T}(w(0, \#))$ .*

*Proof.* Let

$$\begin{aligned} \mathcal{T}(w(0, \#)) &= \langle V, \rightarrow_{\mathcal{T}}, \mathbf{q}, F, \lambda_{\mathcal{T}, \mathbf{H}}, \lambda_{\mathcal{T}} \rangle, \\ \mathcal{G}(w) &= \langle V(w), \rightarrow_{\mathcal{G}}, \mathbf{q}(0, \#), \{\varepsilon\}, \lambda_{\mathcal{G}, \mathbf{H}}, \lambda_{\mathcal{G}} \rangle. \end{aligned}$$

We first claim that, for every  $\alpha \in \Sigma^*$ , we have

$$(19) \quad \mathbf{q}.\alpha = \mathbf{q}(i, a) \implies \mathcal{T}(w(0, \#))_{\mathbf{q}.\alpha} = \mathcal{T}(w(i, a)).$$

To prove this, we argue by induction on  $|\alpha|$ . If  $|\alpha| = 0$ , then the result holds trivially. Let  $\alpha \in \Sigma^*$  be such that  $|\alpha| \geq 1$  and suppose that the result holds for every other shorter word  $\alpha$ . We can write  $\alpha = \beta\gamma$ , with  $\gamma \in \{0, 1\}$ . Let  $\mathbf{q}.\beta = \mathbf{q}(i, a)$ . By induction hypothesis, it follows that  $\mathcal{T}(w(0, \#))_{\mathbf{q}.\beta} = \mathcal{T}(w(i, a))$ . Let  $b_k$  be the principal marker of  $\overline{w}(i, a)$ . By definition of  $\mathcal{G}(w)$ , we have  $\mathbf{q}(0, \#).\beta 0 = \mathbf{q}(i, b)$  and  $\mathbf{q}(0, \#).\beta 1 = \mathbf{q}(k, a)$ . On the other hand, Lemma 3.9 gives that if  $\text{lbf}(w(i, a)) = (w_\ell, b, w_r)$ , then

$\mathcal{T}(w(i, a)) = (\mathcal{T}(w_\ell), \text{reg}(w_\ell) \mid b, \mathcal{T}(w_r))$ , which in turn, by Corollary 5.8, is equivalent to

$$(20) \quad \mathcal{T}(w(i, a)) = (\mathcal{T}(w(i, b)), \text{reg}(w(i, b)) \mid b, \mathcal{T}(w(k, a))).$$

In particular, we conclude that  $\mathcal{T}(w(0, \#))_{q, \beta_0} = \mathcal{T}(w(i, b))$  and  $\mathcal{T}(w)_{q, \beta_1} = \mathcal{T}(w(k, a))$ . It is now enough to notice that, for each pair  $(i, a) \in [0, |\overline{w}|[ \times c_A(\overline{w})$ , the labels of the node  $q(i, a)$  of  $\mathcal{G}(w)$  and the labels of the root of  $\mathcal{T}(w(i, a))$  coincide. In fact, if  $b_k$  is the principal marker of  $\overline{w}(i, a)$ , then the construction of  $\mathcal{G}(w)$  yields the equalities  $\lambda_{\mathcal{G}}(q(i, a)) = b$  and  $\lambda_{\mathcal{G}, \mathbf{H}}(q(i, a)) = \rho_{\mathbf{H}}(\text{reg}(w(i, b)))$ , which, by (20), are precisely the labels of the root of  $\mathcal{T}(w(i, a))$ .  $\square$

Imagine we are given a  $\kappa$ -word and let  $w = a^{\omega+q}$  be one of its representations as a  $\overline{\kappa}$ -term, with  $q$  “very big”. Then, we have  $\overline{w} = 0_0[a_1]^q \#_2$  and so,  $|\overline{w}| = 3$ . Conceptually speaking, such a  $\kappa$ -word involves a “large” number of implicit operations of  $\kappa$  but the length of its representation  $\overline{w}$  in  $\text{Dyck}(A \times \mathbb{N})$  is just 3. Therefore, allowing any representation of  $\kappa$ -words, we would not be able to get meaningful results for the efficiency of the forthcoming algorithms. Thus, it is reasonable to require that all  $\kappa$ -words are presented as  $\kappa$ -terms. We make that assumption from now on.

Consider a  $\kappa$ -term  $w$ . We may assume that  $w$  is given by a tree. For instance, if  $w = (((b^{\omega-1}) \cdot a) \cdot c) \cdot (((a \cdot b) \cdot (a^{\omega-1}))^{\omega-1})$ , then the tree representing  $w$  is depicted in Figure 1. Since from such a tree representation

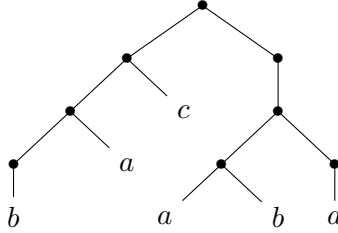


FIGURE 1. The tree representing  $((((b^{\omega-1}) \cdot a) \cdot c) \cdot (((a \cdot b) \cdot (a^{\omega-1}))^{\omega-1}))$ .

we may compute  $\overline{w}$  in linear time, we assume that we are already given  $\overline{w}$ . If the tree representing  $w$  has  $n$  nodes then, following [5], we say that the *length* of  $w$  is  $|w| = n + 1$ . It is clear that  $O(|w|) = O(|\overline{w}|)$ . To actually compute the DRH-graph  $\mathcal{G}(w)$  we essentially need to compute the principal marker of the words  $\overline{w}(i, a)$  as well as the regular parts of  $w(i, a)$ . Almeida and Zeitoun [5] exhibited an algorithm to compute the first occurrences of each letter of a well-parenthesized word  $x$ . Given a word  $x$ ,  $\text{first}(x)$  consists of a list of the first occurrences of each letter in  $x$ . In particular, this computes the principal marker of  $x$ : it is the last entry of the outputted list. Moreover, if  $b_k$  is the principal marker of  $x$ , then the penultimate entry of the list is the principal marker of  $p_b(x)$ , and so on. Hence, this is enough to almost compute  $\mathcal{G}(w)$ . More precisely, the knowledge of  $\text{first}(\overline{w}(i, a))$ , for every pair



It remains to find the labels of the states under  $\lambda_H$ . For that purpose, we observe that the regular part of a pseudoword  $u$  depends deeply on the content of the factors of the form  $\text{lbf}_k(u)$ , which we may compute using Lemma 5.7; and of the cumulative content of  $u$ . Also, it follows from Lemma 3.7 and from Proposition 6.2 that the cumulative content of any pseudoword of the form  $w(i, a)$  is completely determined by the reduct  $\mathcal{G}_R(w)$ . Thus, we may start by computing the cumulative content of  $w(i, a)$  and then compare it with the content of  $\text{lbf}_k(w(i, a))$ , for increasing values of  $k$ . When we achieve an equality, we know what is the regular part of  $w(i, a)$ . Algorithm 1 does that job. We assume that we already have the table described in Lemma 6.3, so that, computing  $c(w(i, a))$  and the principal marker of  $\overline{w}(i, a)$  takes  $O(1)$ -time. Further, we may assume that we are given  $\mathcal{G}_R(w)$ , since we already explained how to get it from the table of Lemma 6.3 in  $O(|w| |c(w)|)$ -time.

```

Require: A  $\bar{\kappa}$ -term  $w$  and  $(i, a) \in [0, |\bar{w}|[ \times c_A(\bar{w})$  (with  $\bar{w}(i, a) \neq \varepsilon$ )
Ensure:  $\text{reg}(w(i, a)) = I$ , if  $\bar{c}(w(i, a)) = \emptyset$  or  $k$  such that  $\text{reg}(w(i, a)) =$ 
     $w(k, a)$ , otherwise
1:  $L \leftarrow \{\}$ ,  $j \leftarrow i$ 
2: while  $j \notin L$  and  $\bar{w}(j, a) \neq \varepsilon$  do
3:    $j \leftarrow \pi_{\mathbb{N}}(\text{principal marker of } \bar{w}(j, a))$   $\triangleright$  So that, if  $\mathbf{q}(j, a).1 \neq \varepsilon$ ,
    then  $\mathbf{q}(j, a) \leftarrow \mathbf{q}(j, a).1$ 
4:    $L \leftarrow L \cup \{j\}$ 
5: end while
6: if  $\bar{w}(j, a) = \varepsilon$  then
7:   return  $I$ 
8: else
9:    $C \leftarrow c(w(j, a))$   $\triangleright$  The set  $C$  is the cumulative content of  $w(i, a)$ 
10:   $k \leftarrow i$ 
11:  while  $c_A(\bar{w}(k, a)) \neq C$  do
12:     $k \leftarrow \pi_{\mathbb{N}}(\text{principal marker of } \bar{w}(k, a))$ 
13:  end while
14:  return  $k$ 
15: end if

```

**Lemma 6.4.** *Algorithm 1 returns  $I$  if and only if  $\tilde{c}(w(i, a)) = \emptyset$ . Otherwise, the value  $k$  outputted is such that  $\text{reg}(w(i, a)) = w(k, a)$ . Moreover, the algorithm runs in linear time, provided we have the knowledge of  $\text{first}(w(i, a))$ .*

*Proof.* By Property (A.3) of a DRH-automaton, and since there is only a finite number of possible states in  $\mathcal{G}_R(w)_{q(i,a)}$ , either there exists  $k \geq 0$  such that  $q(i,a).1^k = \varepsilon$ , or there exists  $\ell > k \geq 0$  such that  $q(i,a).1^k = q(i,a).1^\ell$ . Therefore, the cycle **while** in line 2 does not run forever. If the occurring situation is the former, then  $\vec{c}(\mathcal{G}(w)_{q(i,a)}) = \emptyset$ . On the other hand, by Proposition 6.2, we have  $\mathcal{G}(w)_{q(i,a)} \sim \mathcal{T}(w(i,a))$  which in turn, by Theorem 3.8, implies  $\pi(\mathcal{G}(w)_{q(i,a)}) \mathcal{R} w(i,a)$  modulo DRH. Also, Lemma 3.7 yields  $\vec{c}(w(i,a)) = \vec{c}(\mathcal{G}(w)_{q(i,a)}) = \emptyset$ , and therefore,  $\text{reg}(w(i,a)) = I$ . This is the case where the symbol  $I$  is returned in line 7.

Now, suppose that  $\ell > k \geq 0$  are such that  $q(i,a).1^k = q(i,a).1^\ell$ . Then, the cycle **while** is exited because an index  $j$  is repeated. By Property (A.4), we have a chain of inclusions:  $\lambda(\mathcal{G}(w)_{q(i,a).1^k}) \supseteq \lambda(\mathcal{G}(w)_{q(i,a).1^{k+1}}) \supseteq \dots \supseteq \lambda(\mathcal{G}(w)_{q(i,a).1^\ell})$ . As  $q(i,a).1^k = q(i,a).1^\ell$ , these inclusions are actually equalities, implying that  $k$  is greater than or equal to  $\text{r.ind}(\mathcal{G}(w)_{q(i,a)})$ . Combining again Proposition 6.2, Theorem 3.8 and Lemma 3.7, we may deduce that  $\vec{c}(w(i,a)) = \vec{c}(\mathcal{G}(w)_{q(i,a)}) = \lambda(\mathcal{G}(w)_{q(i,a).1^k})$ , where the last member is precisely  $c(w(j,a))$  provided that  $q(i,a).1^k = q(j,a)$ . Therefore, in line 9 we assign to  $C$  the cumulative content of  $w(i,a)$ . Until now, since we are assuming that we are given all the information about  $\mathcal{G}_R(w)$ , we only spend time  $O(|w|)$ , because that is the number of possible values of  $j$  that may appear in line 2.

Let us prove that, if we get to line 5, then the value  $k$  outputted in line 14 is such that  $\text{reg}(w(i,a)) = w(k,a)$ . We write

$$w(i,a) = \text{lbf}_1(w(i,a)) \cdots \text{lbf}_m(w(i,a))w'_m,$$

for every  $m \geq 1$  (notice that  $\text{lbf}_m(w(i,a))$  is defined for all  $m \geq 1$  because we are assuming that  $\vec{c}(w(i,a)) \neq \emptyset$ ). Then, the regular part of  $w(i,a)$  is given by  $w'_\ell$ , where  $\ell = \min\{m : c(w'_m) = \vec{c}(w(i,a))\}$ . In particular, the projection of  $w'_m$  onto  $\overline{\Omega}_A\text{DRH}$  is not regular, for every  $m < \ell$ . Set  $(c_0, k_0) = (a, i)$  and, for  $m \geq 0$ , let  $(c_{m+1}, k_{m+1})$  be the principal marker of  $\overline{w}(k_m, a)$ . By Corollary 5.8, if  $w(k_m, a)$  is not regular modulo DRH, then we have  $\text{lbf}(w(k_m, a)) = (w(k_m, c_{m+1}), c_{m+1}, w(k_{m+1}, a))$ . Therefore, the equality  $w'_m = w(k_m, a)$  holds, for every  $m \leq \ell$ . Thus, the value  $k$  returned in line 14 is precisely  $k_\ell$ , implying that  $\text{reg}(w(i,a)) = w(k,a)$  as intended.

Since there are only  $O(|\overline{w}|)$  possible values for  $k$  and we are assuming that we already know  $\text{first}(w(i, \#))$  for all  $i \in [0, |\overline{w}|[$ , it follows that lines 8–15 run in time  $O(|\overline{w}|)$ .

Therefore, the overall time complexity of Algorithm 1 is  $O(|w|)$ .  $\square$

So far, we possess all the needed information for computing  $\mathcal{G}(w)$ . Putting all the steps together, we obtain the following.

**Theorem 6.5.** *Given a  $\kappa$ -term  $w$ , it is possible to compute the DRH-graph of  $w$  in time  $O(|w|^2 |c(w)|)$ .  $\square$*

The next question we should answer is how can we decide whether two DRH-graphs  $\mathcal{G}(u)$  and  $\mathcal{G}(v)$  represent the same element of DRH, that is, whether  $\mathcal{G}(u) \sim \mathcal{G}(v)$ . A possible strategy consists in visiting states in both DRH-graphs, comparing their labels (in a certain order). When we find a pair of mismatching labels, we stop, concluding that  $\mathcal{G}(u)$  and  $\mathcal{G}(v)$  are not equivalent. Otherwise, we conclude that they are equivalent after visiting all the states. More precisely, starting in the roots of  $\mathcal{G}(u)$  and  $\mathcal{G}(v)$ , we mark the current states, say  $\mathbf{q}_u \in V(u)$  and  $\mathbf{q}_v \in V(v)$ , as visited, and then repeat the process relatively to the pairs of DRH-automata  $(\mathcal{G}(u)_{\mathbf{q}_u.0}, \mathcal{G}(v)_{\mathbf{q}_v.0})$  and  $(\mathcal{G}(u)_{\mathbf{q}_u.1}, \mathcal{G}(v)_{\mathbf{q}_v.1})$ . For a better understanding of the procedure, we sketch it in Algorithm 2.

---

**Algorithm 2**


---

**Require:** two DRH-graphs  $\mathcal{G}_i = \langle V_i, \rightarrow_i, \mathbf{q}_i, \lambda_{i,H}, \lambda_i \rangle$  ( $i = 1, 2$ )

**Ensure:** logical value of “ $\mathcal{G}_1 \sim \mathcal{G}_2$ ”

```

1: if  $\mathbf{q}_1 = \varepsilon$  then
2:   return logical value of  $\mathbf{q}_2 = \varepsilon$ 
3: else if  $\mathbf{q}_1$  or  $\mathbf{q}_2$  is unvisited then
4:   mark  $\mathbf{q}_1$  and  $\mathbf{q}_2$  as visited
5:   if  $\lambda_{1,H}(\mathbf{q}_1) = \lambda_{2,H}(\mathbf{q}_2)$  and  $\lambda_1(\mathbf{q}_1) = \lambda_2(\mathbf{q}_2)$  then
6:     return logical value of “ $(\mathcal{G}_1)_{\mathbf{q}_1.0} \sim (\mathcal{G}_2)_{\mathbf{q}_2.0}$  and  $(\mathcal{G}_1)_{\mathbf{q}_1.1} \sim$ 
        $(\mathcal{G}_2)_{\mathbf{q}_2.1}$ ”
7:   else
8:     return False
9:   end if
10: else
11:   return logical value of  $(\lambda_{1,H}(\mathbf{q}_1), \lambda_1(\mathbf{q}_1)) = (\lambda_{2,H}(\mathbf{q}_2), \lambda_2(\mathbf{q}_2))$ 
12: end if

```

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**Lemma 6.6.** *Algorithm 2 returns the logical value of “ $\mathcal{G}_1 \sim \mathcal{G}_2$ ” for two input DRH-graphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . Moreover, it runs in time  $O(p \max\{|V_1|, |V_2|\})$ , where  $p$  is such that the word problem modulo  $H$  for any pair of labels  $\lambda_{1,H}(\mathbf{v}_1)$  and  $\lambda_{2,H}(\mathbf{v}_2)$  (with  $\mathbf{v}_1 \in V_1$  and  $\mathbf{v}_2 \in V_2$ ) may be solved in time  $O(p)$ .*

*Proof.* The correctness follows straightforwardly from the definition of the relation  $\sim$ . On the other hand, it runs in time  $O(p \max\{|V_1|, |V_2|\})$ , since each call of the algorithm takes time  $O(p)$  (line 5) and each pair of states of the form  $(\mathbf{q}_1.\alpha, \mathbf{q}_2.\alpha)$  is visited exactly once.  $\square$

Given  $\bar{\kappa}$ -terms  $u$  and  $v$ , we use  $p(u, v)$  to denote a function depending on some parameters associated with  $u$  and  $v$  (that may be, for instance,  $|u|$ ,  $|v|$  or  $c(u)$ ,  $c(v)$ ) and such that, the time for solving the word problem over  $H$  for any pair of factors of the form  $u(i, a)$  and  $v(j, b)$  is in  $O(p(u, v))$ . Observe that the time to transform an expression of the form  $u(i, a)$  into a  $\bar{\kappa}$ -term should be taken into account. Furthermore, such a function  $p(u, v)$  is not

unique, but the results are valid for any such function. Then, summing up the time complexities of all the intermediate steps considered above, we have just proved the following result.

**Theorem 6.7.** *Let  $H$  be a pseudovariety of groups with decidable  $\kappa$ -word problem, and let  $u$  and  $v$  be  $\kappa$ -terms. Then, the equality of the pseudowords represented by  $u$  and  $v$  over  $\text{DRH}$  can be tested in time  $O((p(u, v) + m)m |A|)$ , where  $m = \max\{|u|, |v|\}$ .  $\square$*

Observe that, in general, the complexity of an algorithm for solving the  $\kappa$ -word problem over  $H$  should depend on the length of the intervening  $\bar{\kappa}$ -terms. It is not hard to see that the length of the factors  $w(i, a)$  grows quadratically on  $|w|$  (we prove it below in Corollary 7.3). Hence, it is expected that, at least in most of the cases,  $m$  belongs to  $O(p(u, v))$ . Consequently, the overall time complexity stated in Theorem 6.7 becomes  $O(p(u, v)m |A|)$ . Since we are doing the same approach as in [5], this result is somehow the expected one. Roughly speaking, this may be interpreted as the time complexity of solving the word problem in  $R$ , together with a word problem in  $H$  for each state, that is, for each  $\text{DRH}$ -factor of the involved pseudowords (recall Lemmas 2.8 and 4.2).

Just as a complement, we mention that another possible approach would be to transform the  $\text{DRH}$ -graph  $\mathcal{G}(w)$  in an automaton in the classical sense, say  $\mathcal{G}'(w)$ , recognizing the language  $\mathcal{L}(w)$ . That is easily done (time linear on the number of states), by moving the labels of a state to the arrows leaving it. More precisely, the automaton  $\mathcal{G}'(w)$  shares the set of states with  $\mathcal{G}(w)$  and each non terminal state  $q(i, a)$  has two transitions:

$$\begin{aligned} q(i, a).(0, \lambda_H(q(i, a)), \lambda(q(i, a))) &= q(i, 1).0 \\ q(i, a).(1, I, \lambda(q(i, a))) &= q(i, a).1. \end{aligned}$$

Then, we could use the results in the literature in order to minimize the automaton, obtaining a unique automaton representing each  $\mathcal{R}$ -class of the semigroup  $(\bar{\Omega}_A \text{DRH})^I$ . The unique issue in that approach is that the algorithms are usually prepared to deal with alphabets whose members may be compared in constant time. Hence, we should previously prepare the input automaton by renaming the subset of the alphabet  $\Sigma \times (\bar{\Omega}_A H)^I \times A$ , in which the labels of transitions are being considered. Let  $p(u, v)$  and  $m$  have the same meaning as in Theorem 6.7. Since, a priori, we do not possess any information about the possible values for  $\lambda_H$ , that would take  $O(p(u, v)(m |A|)^2)$ -time (each time we rename an element in  $(\bar{\Omega}_A H)^I$  we should first verify whether we already encountered another element with the same value over  $H$ ). Thereafter, we could use the linear time algorithm presented in [6], which works for this kind of automaton. Thus, a rough upper bound for the complexity spent using this method is  $O(p(u, v)m^2 |A|^2)$ , which although a bit worse, is still polynomial.

The following result gives us a family of pseudovarieties of the form DRH with decidable  $\kappa$ -word problem. It is a consequence of the fact that the free group is residually in  $G_p$ .

**Corollary 6.8.** *Let  $p$  be a prime number. If  $H \supseteq G_p$  is a pseudovariety of groups, then the pseudovariety DRH has decidable  $\kappa$ -word problem.*  $\square$

## 7. AN APPLICATION: SOLVING THE WORD PROBLEM OVER DRG

Let us illustrate the previous results by considering the particular case of the pseudovariety DRG. By Theorem 6.7, the time complexity of our procedure for testing identities of  $\kappa$ -terms modulo DRG depends on a certain parameter  $p(-, -)$ . In order to discover that parameter, we should first analyze the (length of the) projection onto  $\Omega_A^\kappa G = FG_A$  of the elements of the form  $w(i, a)$ , where  $w$  is a  $\kappa$ -term.

Consider the alphabets  $B_1 = (A \times \mathbb{N}) \uplus \{[-^1, ]^{-1}\}$  and  $B_2 = (A \times \mathbb{N}) \uplus \{[-^1, [-^2, ]^{-1}, ]^{-2}\}$ . Let  $x$  be a well-parenthesized word over  $B_2$ . The *expansion* of  $x$  is the well-parenthesized word  $\exp(x)$  obtained by successively applying the rewriting rule  $[-^2y]^{-2} \rightarrow [-^1y]^{-1}[-^1y]^{-1}$ , whenever  $y$  is a well-parenthesized word. It is clear that  $\text{om}(x)$  and  $\text{om}(\exp(x))$  represent the same  $\kappa$ -word and that  $x$  is a well-parenthesized word over  $B_1$ . Further, we have the following.

**Lemma 7.1.** *Let  $x$  be a nonempty well-parenthesized word over  $B_1$  and  $i \in c_{\mathbb{N}}(x)$ . Then,  $t_i(x)$  is a well-parenthesized word over  $B_2$  and  $|\exp(t_i(x))| \leq \frac{1}{2}(|x|^2 + 2|x| - 3)$ . Moreover, this upper bound is tight for all odd values of  $|x|$ .*

*Proof.* The fact that  $t_i(x)$  is a well-parenthesized word over  $B_2$  follows immediately from the definition of  $t_i$ . To prove the inequality, we proceed by induction on  $|x|$ . If  $x = a_i$ , then  $t_i(x)$  is the empty word and so, the result holds. Let  $x$  be a well-parenthesized word over  $B_1$  such that  $|x| = n$ . The inequality holds clearly, unless  $x$  is of the form  $x = [-^1y]^{-1}z$ , with  $y$  and  $z$  well-parenthesized words over  $B_1$ ,  $y$  nonempty and  $i \in c_{\mathbb{N}}(y)$ . In that case, we have  $t_i(x) = t_i(y)[-^2y]^{-2}z$ . Using induction hypothesis on  $y$ , one may deduce that  $|\exp(t_i(y))| \leq \frac{1}{2}(|y|^2 + 2|y| - 3)$ . Finally, let  $\vec{x} = (a_1, \varepsilon, \varepsilon, \dots)$ ,  $\vec{y} = (\varepsilon, \varepsilon, \dots)$ ,  $\vec{q} = (-1, -1, \dots)$ , and  $u_{2n+1} = \mu_n(\vec{x}, \vec{y}, \vec{q})$  (recall the notation used in Lemma 5.4). Then,  $u_{2n+1}$  is a well-parenthesized word over  $B_1$  of

length  $2n + 1$ . Moreover, using Lemma 5.4, we may compute

$$\begin{aligned}
|\exp(\mathbf{t}_1(u_{2n+1}))| &= |\exp(\mathbf{t}_1(\mu_0(\vec{x}, \vec{y}, \vec{q})) \cdot \xi_0(\vec{x}, \vec{y}, \vec{q}) \cdot \xi_1(\vec{x}, \vec{y}, \vec{q}) \cdots \xi_{n-1}(\vec{x}, \vec{y}, \vec{q}))| \\
&= |\exp([-^2\mu_0(\vec{x}, \vec{y}, \vec{q})]^{-2} \cdots [-^2\mu_{n-1}(\vec{x}, \vec{y}, \vec{q})]^{-2})| \\
&= \sum_{k=0}^{n-1} 2(|\mu_k(\vec{x}, \vec{y}, \vec{q})| + 2) \\
&= 2n^2 + 4n \quad \text{because } |\mu_k(\vec{x}, \vec{y}, \vec{q})| = 2k + 1 \\
&= \frac{1}{2}(|u_{2n+1}|^2 + 2|u_{2n+1}| - 3)
\end{aligned}$$

and the result follows.  $\square$

Also, as a straightforward consequence of the definition of  $\mathbf{p}_a$ , the following holds.

**Lemma 7.2.** *Let  $x$  be a nonempty well-parenthesized word over  $B_1$  and  $a \in A$ . Then,  $\mathbf{p}_a(x)$  is also a well-parenthesized word over  $B_1$  and  $|\exp(\mathbf{p}_a(x))| = |\mathbf{p}_a(x)| \leq |x|$ .*  $\square$

Given a well-parenthesized word  $x$  over  $B_2$ , we define the *linearization over  $A$*  of  $x$  to be the word  $\text{lin}(x)$  over the alphabet  $A \uplus A^{-1}$  obtained by applying the rewriting rules  $[-^1a_i]^{-1} \rightarrow a^{-1}$ ,  $[-^1yz]^{-1} \rightarrow [-^1z]^{-1}[-^1y]^{-1}$  and  $[-^2y]^{-2} \rightarrow [-^1y]^{-1}[-^1y]^{-1}$  to  $x$  (with  $a_i \in c(x)$  and  $y, z$  well-parenthesized words). It is easy to see that  $\text{lin}(x) = \text{lin}(\exp(x))$  and that if  $x$  is a well-parenthesized word over  $B_1$ , then  $O(|\text{lin}(x)|) = O(|x|)$ . Consequently, we have the next result.

**Corollary 7.3.** *Let  $w$  be an  $\kappa$ -term and  $(i, a) \in [0, |\overline{w}|[ \times c_A(\overline{w})$ . Then,  $|\text{lin}(\overline{w}(i, a))|$  belongs to  $O(|w|^2)$ .*  $\square$

Now, we wish to compute  $\text{lin}(x)$ , for a given well-parenthesized word over  $B_2$ . Recall the tree representation of  $\kappa$ -terms exemplified in Figure 1. We may recover, in linear time, such a tree representation for  $\text{om}(x)$ , for a well-parenthesized word  $x$  over  $B_1$ . Furthermore, if we are given a well-parenthesized word over  $B_2$ , we may compute, also in linear time, a tree representation for  $\text{om}(\exp(x))$ . That amounts to, whenever we have a factor of the form  $[-^2y]^{-2}$  in  $x$ , to include twice a subtree representing  $[-^1y]^{-1}$ .

On the other hand, since solving the word problem in  $\text{FG}_A$  (for words written over the alphabet  $A \cup A^{-1}$ ) is a linear issue in the size of the input, by Corollary 7.3, we may take  $p(u, v) = \max\{|u|^2, |v|^2\}$ . Thus, we have proved the following.

**Proposition 7.4.** *The  $\kappa$ -word problem over DRG is decidable in  $O(m^3 |A|)$ -time, where  $m$  is the maximum length of the inputs.*  $\square$

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